17T7 is a Galois group over the rationals

Edgar Costa (MIT)

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Slides available at edgarcosta.org
Joint work with:
Raymond van Bommel, Noam Elkies, Timo Keller, Samuel Schiavone, and John Voight

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This is also nicely organized by Klüners–Malle (galoisdb.math.upb.de).

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giving

$$1 \to \mathrm{SL}_2(\mathbb{F}_{16}) \to 17T7 \to C_2 \to 1.$$

Main theorem

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Question

How does one construct such field?

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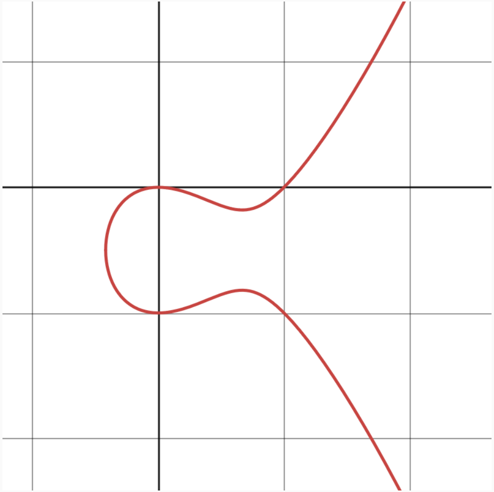

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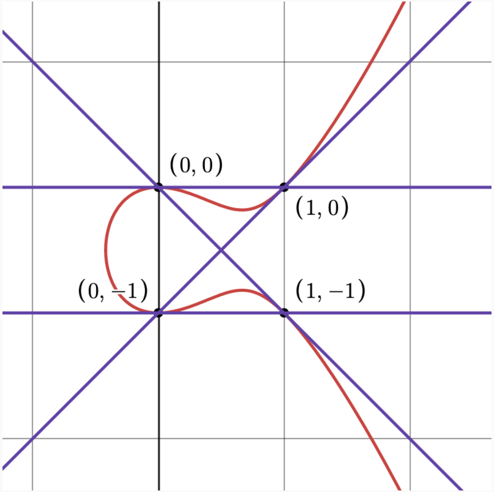

$E\colon y^2 + y = x^3 - x^2$ and $(0,0) \in E[5](\mathbb{Q})$.

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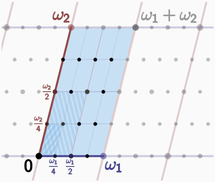

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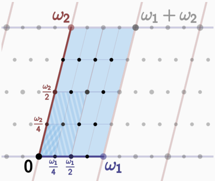

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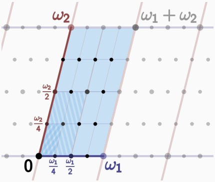

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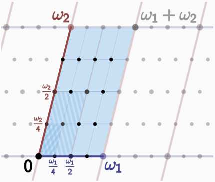

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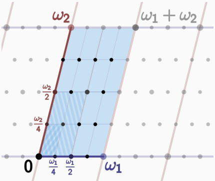

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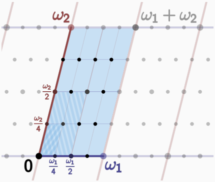

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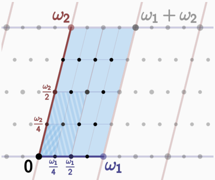

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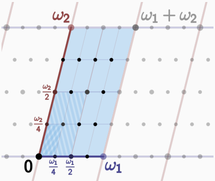

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Slogan

In number theory, maximal entropy is the norm.

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$$E\colon y^2 = x^3 - x \quad (32.a3)$$

has the additional symmetry $(x,y) \mapsto (-x, iy)$ giving $R = \mathbb{Z}[i] \subset K = \mathbb{Q}(i)$ acting by endomorphisms of E, written $R \hookrightarrow \mathbf{End}(E_K)$. But then we don't just have

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Slogan

Additional symmetries (endomorphisms) must be respected.

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$$P + Q + R \sim 0$$

In general, we can think about adding tuples of g-points.

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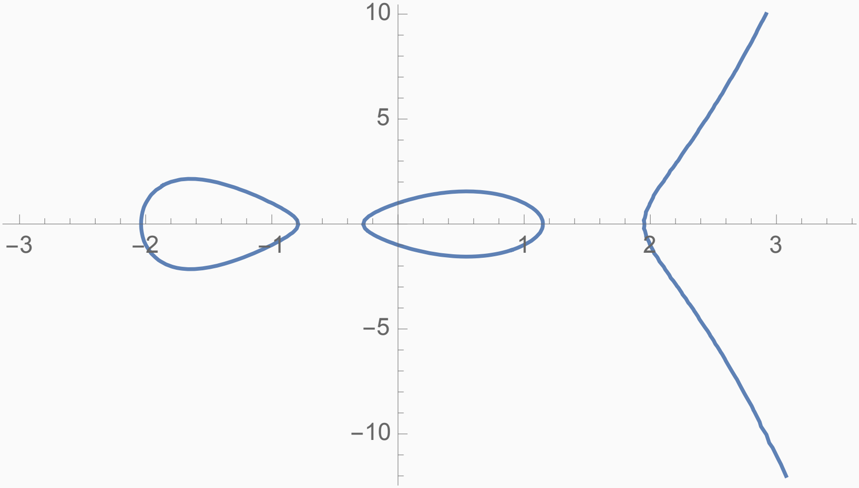

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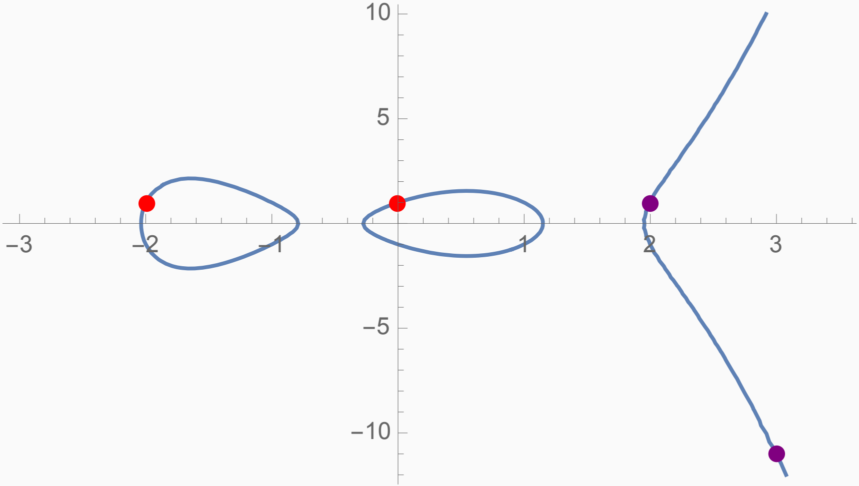

$D_1 := (-2, 1) + (0, 1)$

$D_2 := (2, 1) + (3, -11)$

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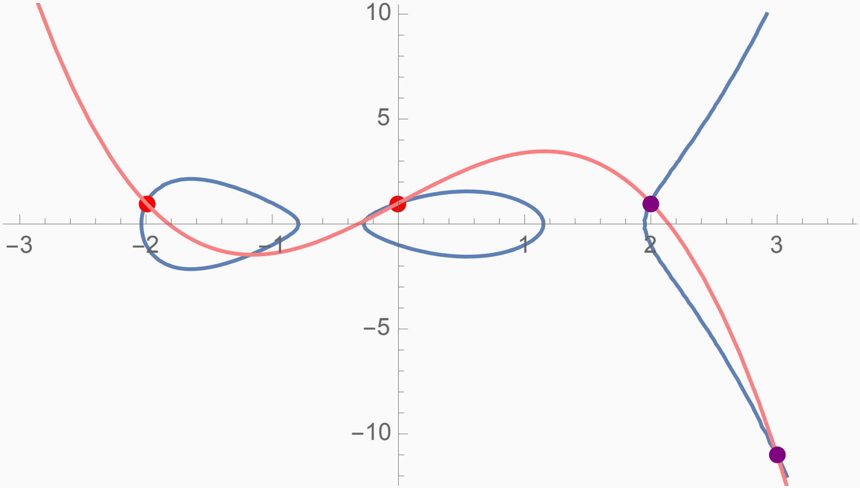

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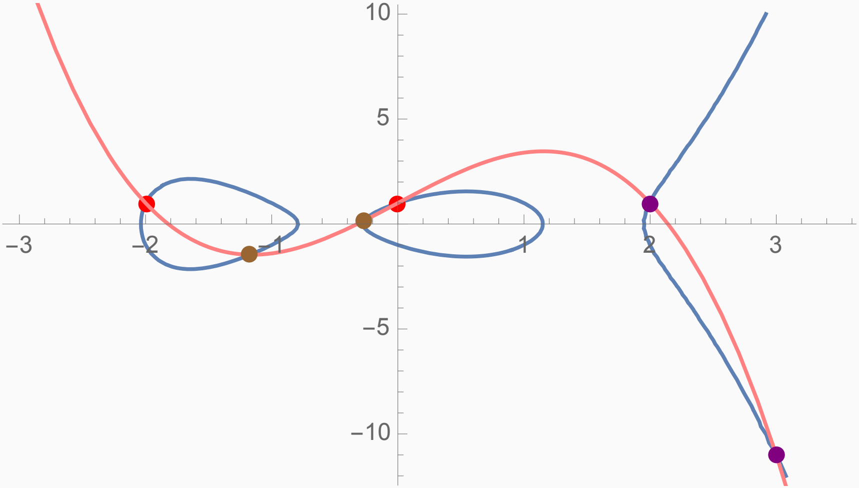

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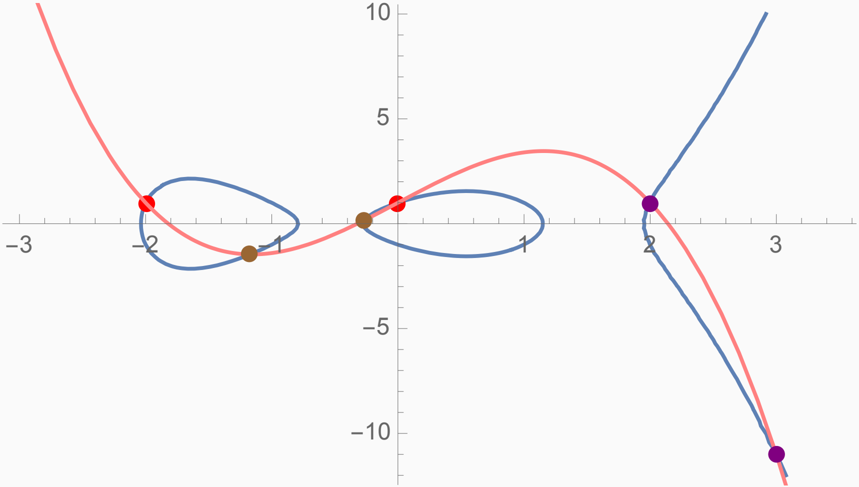

$$(\textcolor{red}{\bullet} + \textcolor{red}{\bullet}) + (\textcolor{purple}{\bullet} + \textcolor{purple}{\bullet}) + (\textcolor{brown}{\bullet} + \textcolor{brown}{\bullet}) = 0$$

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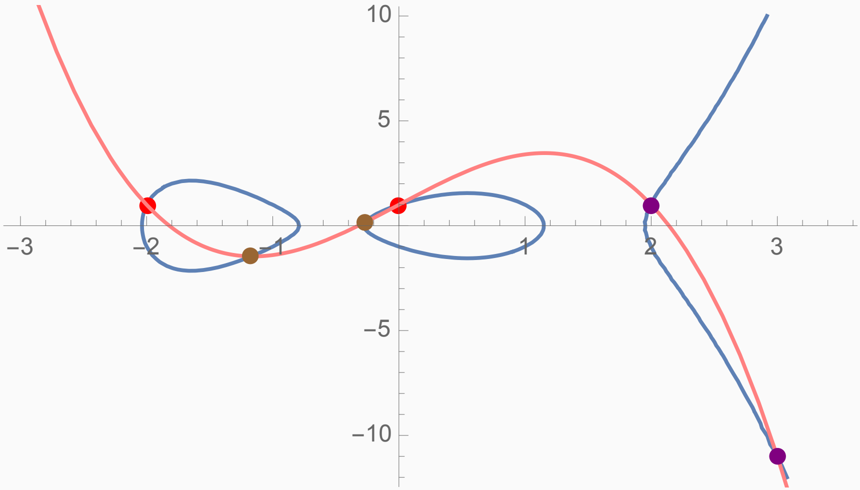

$$D_3 := \left(\frac{-\sqrt{209}-23}{32}, \frac{-115\sqrt{209}-1333}{2048}\right) + \left(\frac{\sqrt{209}-23}{32}, \frac{115\sqrt{209}-1333}{2048}\right)$$

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We can again cut down on the image of Galois by additional endomorphisms.

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A Hilbert modular form over $F = \mathbb{Q}(\sqrt{3})$ with *Galois alignment*: 2.2.12.1-578.1-d.

Classical Modular forms

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A (classical) modular form f of weight k on $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$, is a holomorphic function defined on the the upper half plane $\mathfrak{h} := \{z : \Im(z) > 0\}$ which satisfies the transformation property

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If $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$, then $f(z) = f(z+1)$ and f has a Fourier expansion

$$f(z) = \sum_{n\geq 0} a_n q^n, \quad q = e^{2\pi i z}.$$

If $a_0 = 0$ and $a_1 = 1$, then f is known as a cusps form.

www.lmfdb.org/ModularForm/GL2/Q/holomorphic/

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To an eigenform

$$f = \sum_{n \geq 1} a_n q^n \in S_2^{\mathrm{new}}(\Gamma_0(N)) \qquad a_n \in K_f := \mathbb{Q}(a_1, a_2, a_3, \dots)$$

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and enables one compute the period matrix of A_f to any desired precision.

Hilbert modular forms to geometry: Eichler–Shimura construction

Replacing $SL_2(\mathbb{Z})$ with $GL_2^+(\mathbb{Z}_F)$, where $F \subset \mathbb{R}$, gives us Hilbert modular forms.

These follow similar transformation rules and also come with Fourier expansions

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$$H^1(A_f, \mathbb{Q}) \otimes_{K_f} H^1(A_f', \mathbb{Q}) = H^2(X_0(\mathfrak{N})[f], \mathbb{Q})$$

as K_f-Hodge structures.

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Recovering an abelian variety from its L-function via BSD

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For a quadratic character χ of signature ss', we have

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Dembélé showcased such approach for reconstructing elliptic curves over F.

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We construct $A_f(\mathbb{C}) \simeq \mathbb{C}^g/\Lambda_f$ by taking

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Theorem

We have $P \in \mathbb{Q}[t]$ and has Galois group 17T7.

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&- 1148273598471179728781481033200461057071613065513809470959738416912t^{12} \\
&- 1814416503358004575011887633363669311563353153960463604533351275745379344187064t^{11} \\
&- 1770863661928284803713567743362051511470304070815670165425643871240590168197004899614562992t^{10} \\
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github.com/edgarcosta/EichlerShimuraHMF

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CPU/Human time: ??

Curve

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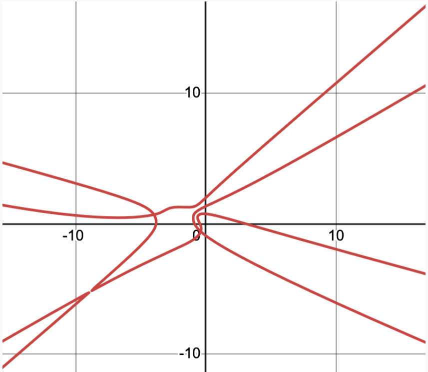
With refined $\tau(A_f)$ we were able to numerically reconstruct a genus 4 curve in $\mathbb{P}^3$:

$$-8x^2 + 8xy + 17y^2 - 34xz - 2yz - 28z^2 - 10xw - 9yw - 18zw + 2w^2 = 0$$

$$4x^3 - 6x^2y - 6xy^2 + 12x^2z + 6xyz + 24y^2z - 12xz^2 - 24z^3 + 2x^2w$$
$$+7xyw + 4y^2w + 4xzw - 13yzw - 8z^2w - 20xw^2 - 3zw^2 - 12w^3 = 0$$



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The effective inverse Galois problem holds for the group 17T7.

The polynomial

$$f(x) = x^{17} - 2x^{16} + 12x^{15} - 28x^{14} + 60x^{13} - 160x^{12} + 200x^{11} - 500x^{10} + 705x^{9} \\ - 886x^{8} + 2024x^{7} - 604x^{6} + 2146x^{5} + 80x^{4} - 1376x^{3} - 496x^{2} - 1013x - 490$$

has $\mathrm{Gal}(f) \simeq 17\mathrm{T}7 \simeq \mathrm{SL}_2(\mathbb{F}_{16}) \rtimes C_2$.

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- Are there infinitely many? Who knows