Eichler–Shimura Construction for Hilbert Modular Forms

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Slides available at edgarcosta.org

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Classical modular forms

To an eigenform

$$f = \sum_{n \geq 1} a_n q^n \in S_2^{\text{new}}(\Gamma_0(N)) \qquad a_n \in K_f := \mathbb{Q}(a_1, a_2, a_3, \dots)$$

one can attach an abelian variety $A_f/\mathbb{Q}$ such that

$$\dim A_f = \deg K_f$$

$$L(A_f, s) = \prod_{\sigma : K_f \hookrightarrow \mathbb{C}} L(f^\sigma, s)$$

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This construction can be made explicit via the Jacobian of $X_0(N)$

$$J_0(N) \sim \bigoplus_{M|N} \bigoplus_{f \in G_\mathbb{Q} \backslash S_2^{\mathrm{new}}(\Gamma_0(M))} A_f^{\sigma_0(N/M)}$$

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and enables one compute the period matrix of A_f to any desired precision.

Hilbert modular forms

Replacing $SL_2(\mathbb{Z})$ with $GL_2^+(\mathbb{Z}_F)$, where $F \subset \mathbb{R}$, gives us Hilbert modular forms.

These also come with Fourier expansions

$$f = a_0 + \sum_{\nu \in \mathcal{D}^{-1}_{>0}} a_\nu q^\nu$$

seen as differential forms in the modular variety $X_0(\mathfrak{N})$ of dimension $\deg F$.

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To an eigenform $f \in S_2^{\text{new}}(\Gamma_0(\mathfrak{N}))$ we also expect the existence of A_f/F, such that:

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However, for $F \neq \mathbb{Q}$, we no longer have a Jacobian to work with!

Eichler–Shimura relations for real quadratic fields

Theorem (Oda)

Let F be a totally real quadratic field with trivial narrow class group.

Let $f \in S_2^{\mathrm{new}}(\Gamma_0(\mathfrak{N}))$ be an eigenform with eigenvalue field K_f.

There exists abelian varieties $A_f/\mathbb{C}$ and $A'_f/\mathbb{C}$ of dimension $g = \deg K_f$ such that

$$H^1(A_f, \mathbb{Q}) \otimes_{K_f} H^1(A'_f, \mathbb{Q}) = H^2(X_0(\mathfrak{N})[f], \mathbb{Q})$$

as K_f-Hodge structures.

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The construction is not explicit, and the fields of definition are unknown.

Recovering an abelian variety from its L-function A

Nonetheless, Oda gives an explicit formula for their periods $\tau(A_f), \tau(A_f') \in \mathfrak{h}^g$.

$$\tau(A_f) = \left\{ \frac{\Omega^{+-}}{\Omega^{++}}(f^\sigma) \right\}_{\sigma : K_f \hookrightarrow \mathbb{C}} \qquad \tau(A_f') = \left\{ \frac{\Omega^{-+}}{\Omega^{++}}(f^\sigma) \right\}_{\sigma : K_f \hookrightarrow \mathbb{C}}$$

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OBSD conjecture

For a quadratic character χ of signature ss', we have

$$\alpha_\chi \Omega^{ss'}(f^\sigma) = -4\pi^2 \sqrt{\operatorname{disc} F} G(\overline{\chi}) L(f^\sigma \otimes \chi, 1) \qquad \text{for some } \alpha_\chi \in \mathbb{Z}_F.$$

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By computing $L(f^\sigma \otimes \chi, 1)$ for several χ, we can guess the periods $\tau(A_f)$ and $\tau(A'_f)$.

Dembélé showcased such approach for elliptic curves.

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The goal is to industrialize and generalize this approach.

Inverse Galois problem for $\mathrm{SL}_2(\mathbb{F}_{16}) \rtimes C_2 \simeq$ 17T7

Theorem (van Bommel–C–Elkies–Derickx–Keller–Schiavon–Voight)

There is a finite Galois extension $L/\mathbb{Q}$ such that $\mathrm{Gal}(L/\mathbb{Q}) \simeq \mathrm{SL}_2(\mathbb{F}_{16}) \rtimes C_2$.

The proof is non-constructive.

It relies on a modulo 2 Galois representation attached an Hilbert modular form f.

We aim to make the proof explicit by "reconstructing" the abelian fourfold A_f/K.