Hypergeometric motives in the LMFDB

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Hypergeometric datum

A **hypergeometric datum** of degree r is defined by two disjoint tuples

$$(\alpha_1, \ldots, \alpha_r), (\beta_1, \ldots, \beta_r) \text{ over } \mathbb{Q} \cap [0,1)$$

which are each **balanced**: the multiplicity of any reduced fraction depends only on its denominator. For example

$$\alpha = (\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{4}),\ \beta = (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3}).$$

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This datum defines a family of hypergeometric motives $M_z^{\alpha,\beta}$ with $z \in \mathbb{Q} \setminus \{0, 1\}$, and a family of degree r L-functions:

$$L(M_z^{\alpha,\beta}, s) = \prod_p F_p(M_z^{\alpha,\beta}, p^{-s}) = \sum_{n \geq 1} \frac{a_n}{n^s}$$

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L-functions of hypergeometric motives

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The primes p of **bad reduction** have the following forms.

- p is **wild** if $v_p(\gamma) < 0$ for some $\gamma \in \alpha \cup \beta$ (e.g., 2 and 3 in our example).

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- p is **wild** if $v_p(\gamma) < 0$ for some $\gamma \in \alpha \cup \beta$ (e.g., 2 and 3 in our example).
- p is **tame** if it is not wild, and either $v_p(z) \neq 0$ or $v_p(z-1) \neq 0$.

If one completes the L-function

$$\Lambda(s) := N^{s/2} \cdot \Gamma_{\alpha,\beta}(s) \cdot L\left(M_z^{\alpha,\beta}, s\right)$$

We expect Λ to satisfy the functional equation

$$\Lambda(s) = \epsilon \Lambda(w + 1 - s).$$

To experimentally check this, one needs to know $a_n \leq B$, where $B \in O(\sqrt{N})$.

The Good, the Tame and the Wild

$$L(M_z^{\alpha,\beta}, s) = \prod_p F_p(M_z^{\alpha,\beta}, p^{-s}) = \sum_{n\geq 1} \frac{a_n}{n^s} = L_{\text{good}}(s) \cdot L_{\text{tame}}(s) \cdot L_{\text{wild}}(s)$$

For p, a good prime, i.e., neither wild nor tame, F_p, may be recovered from a trace formula of the shape

$$H_q\left(\begin{matrix}\alpha \\ \beta\end{matrix}\middle| z\right) := \frac{1}{1-q} \sum_{m=0}^{q-2} \pm p^{\xi(m)} \left(\prod_{j=1}^{r} \frac{(\alpha_j)_m^*}{(\beta_j)_m^*}\right) [z]^m,$$

where $(\gamma)_m^*$ is a p-adic variant of $(\gamma)_m = \gamma(\gamma+1)\cdots(\gamma+m-1)$.

Theorem (CKR20, CKR24)

The complexity of computing a_p for good $p \leq X$ is $O(X)$ modulo log factors.

There is a recipe for F_p at the tame primes.
We do not yet have formulas for F_p at the wild primes.

The Good, the Tame and the Wild

$$\Lambda(s) := N^{s/2}\Gamma_{\alpha,\beta}(s)L(M_z^{\alpha,\beta}, s) = (\Gamma_{\alpha,\beta}(s)\cdot L_{\text{good}})\cdot(N_{\text{tame}}^{s/2}\cdot L_{\text{tame}}(s))\cdot(N_{\text{wild}}^{s/2}\cdot L_{\text{wild}}(s))$$
$$= \Lambda_{\text{good}}(s)\cdot\Lambda_{\text{tame}}(s)\cdot\Lambda_{\text{wild}}(s)$$

- $\Lambda_{\text{good}}(s)$ ✓
  Using the average polynomial time algorithm for a_p [CKR20, CKR24]
  Trace formula for a_{p^i} for $i > 1$
- $\Lambda_{\text{tame}}(s)$ ✓
  There is a recipe for N_{tame} [Roberts–Rodriguez Villegas]
- $\Lambda_{\text{wild}}(s)$ 😕
  No formula or recipe is known.
  But the theory is forming... 😶‍🌫️

Wild primes

David Roberts has a framework to reduce the computational problem of a motive with several wild primes to several motives, each with a single wild prime.

We spent most of last week's workshop gathering data for motives with a single wild prime.

- Using functional equation to deduce wild data.
- Studying how N_{wild} varies with z in certain families (more details later).

L-functions with unknown invariants

By taking inverse Mellin transforms one can convert

$$\Lambda(s) = \epsilon\Lambda(w+1-s) \quad \text{into} \quad \Theta(t) = \epsilon \cdot t^{-w}\Theta(1/t)$$

where

$$\Theta(t) := \sum_{n\geq 1} a_n \phi(nt/\sqrt{N}) \quad \text{and} \quad \phi(t) \in O\big(e^{-rt^{2/r}}\big).$$

One may approximate $\Theta(t)$, hence, the functional equation, to a desired precision by truncating the series expansion of $\Theta(t)$ involving the first $O(\sqrt{N})$ Dirichlet coefficients.

$$\widetilde{\Theta}(t) \approx \epsilon \cdot t^{-w}\widetilde{\Theta}(1/t) \quad \text{where} \quad \widetilde{\Theta}(t) := \sum_{n\leq B} a_n \phi(nt/\sqrt{N})$$

Approximate functional equation

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If one guesses the ϵ, then the functional equation at a point t_0 becomes

$$f_{t_0}(c) \approx 0, \quad \text{with} \quad f_{t_0} \in \mathbb{R}[c_1, \ldots, c_d]$$

where c are the coefficients of the unknown Euler factors.

In other words, we are trying to solve

$$0 \overset{?}{\approx} \min_{c \in \triangle \cap \mathbb{Z}^d} |f_{t_0}(x)|, \quad \text{where} \quad \triangle \text{ is a polytope.}$$

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One can solve this problem in several ways

- Search: looping over all the possible unknown Euler factors. 🤠

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- Minimize: solve the mixed-integer programming problem. 🤔
- Linearize: compute several f_{t_i} and treat each monomial as an unknown. 🤯

Trinomial motives

We consider a sequence of hypergeometric families where we can readily compute conductors.

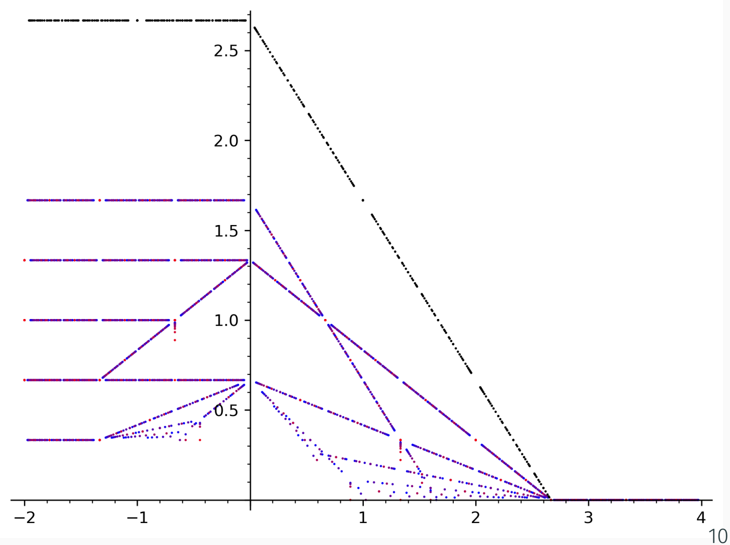
- Fix p, $w > 0$ and $0 \leq r < w$; let $m > 0$ vary, prime to p.
- Set $a = mp^w$, $b = mp^r$ and $c = m(p^w - p^r)$.
- Hypergeometric motive with γ-vector $[-a, b, c]$ is weight zero, with model

  $$a^a x^b (1-x)^c - b^b c^c z.$$

  Substitute $z = up^k$ to get number field K, decompose $K \otimes \mathbb{Q}_p$ as $K_1 \oplus \cdots \oplus K_n$ with residue degrees $f_1, \ldots, f_n$. Compute $\alpha = v_p(\Delta(K)) - a + \sum f_i$, the Swan conductor of K at p.
- Plot $(k/c, \alpha/c)$.
- Varying m corresponds to substituting fractional powers of p.

Trinomial picture (p=2, w=2, r=0)

- x-axis is k/c, where $k = v_p(z)$.
- y-axis is α/c, where α is the Swan conductor at p.
- Top line is the "ramp," where $(k, p) = 1$.



Wild Euler factors of $[1,2,4]$-families

[1,2,4]-families

Hypergeometric families with $\alpha_i, \beta_j \in \{1, 1/2, 1/4, 3/4\}$.

- The only wild prime is 2.
- Assume $k = v_2(z) \neq 0$ and $\sigma(k)$ denote the unscaled ramp as a function of k. Then $\sigma(k)$ is a conjectured upper bound on $c_2 = v_2(N)$.
- Let n_1 denote the number of 1s in α or β, whichever is higher.

Kloosterman polynomials

For each $n \geq 1$, consider the hypergeometric family defined by the n-tuples $\alpha = (1/2, \ldots, 1/2), \beta = (1, \ldots, 1)$. Let $z = 2^{2n}$. The Euler factor at 2 of the motive $M_z^{\alpha,\beta}$ is the Kloosterman polynomial of degree $n_1 = n$.

Predictions and data

Let $M_z^{\alpha,\beta}$ be a hypergeometric motive in a $[1,2,4]$-family, and $k = v_2(z)$.

Predictions about $L_2(T)$ and c_2 (Erasing principle + degenerations at $0, \infty$)

- If $2 \nmid k$, then $L_2(T) = 1$ and $c_2 = \sigma(k)$.
- If $2 \mid k$, then
  1. At bottom of ramp, either $L_2(T)$ is the Kloosterman polynomial of degree n_1 with a 2-power twist factor related to the weight, or $c_2 = 0$.
  2. On low plateau, $L_2(T) = 1$ or a product of at most two polys of the form $1 - 2^i T$.
  3. On high plateau, $L_2(T) = 1$ or of the form $1 - 2^i T$.

Computed data for $M_z^{\alpha,\beta}$ with degree ≤ 7 and $z = \pm 2^k$ for different positions k wrt the ramp. They match predictions. A few exceptions possibly due to: poles, $c_2 = 0$.

Example

$\alpha = (1, 1, \frac{1}{2}, \frac{1}{2}), \beta = (\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}), z = 2^{-8}$. Predicted $L_2(T)$ is $1 + T + 2T^2$.
It turns out that $c_2 = 0$ and $L_2(T) = (1 + T + 2T^2)^2$.