Effective computations of Hasse–Weil zeta functions

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Variation of Néron-Severi ranks of K3 surfaces

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Setup

- k a number field;
- X a K3 surface over k;
- $\mathfrak{p}$ a prime of k where X has good reduction $X_{\mathfrak{p}}$;
- $\mathrm{NS}(\bullet) := \mathrm{Pic}(\bullet)/\mathrm{Pic}_0(\bullet)$, the Néron-Severi group;
  $\rightsquigarrow$ a $\mathbb{Z}$ lattice geometrically associated to the surface $\bullet$
- $\rho(\bullet) := \mathrm{rk}(\mathrm{NS}(\bullet))$, the arithmetic/geometric Picard number of $\bullet$.
  $\rightsquigarrow$ $\rho(\bullet)$ the rank of lattice mentioned above.

Question

- How are the geometric Picard numbers, $\rho(\overline{X})$ and $\rho(\overline{X}_{\mathfrak{p}})$, related?
- How does the geometric Picard number behaves under reduction modulo $\mathfrak{p}$?

Some observations

Question

- How are the geometric Picard numbers, $\rho(\overline{X})$ and $\rho(\overline{X}_{\mathfrak{p}})$, related?
- How does the geometric Picard number behaves under reduction modulo $\mathfrak{p}$?

- There is a natural specialization homomorphism

$$s_{\mathfrak{p}} : \mathrm{NS}(\overline{X}) \hookrightarrow \mathrm{NS}(\overline{X}_{\mathfrak{p}}).$$

  Thus, $\rho(\overline{X}) \leq \rho(\overline{X}_{\mathfrak{p}})$.
- $1 \leq \rho(X) \leq 20$
- $2 \leq \rho(\overline{X}_{\mathfrak{p}}) \leq 22$, and $\rho(\overline{X}_{\mathfrak{p}})$ is always even.
- Computing $\rho(\overline{X})$ is a hard problem.
- We can compute $\rho(\overline{X}_{\mathfrak{p}})$ by counting points on $X_{\mathfrak{p}}$.

Computing $\rho(\overline{X}_{\mathfrak{p}})$

$$Z_{X_{\mathfrak{p}}}(T) := \exp\left(\sum_{i=1}^{\infty} \frac{\#X_{\mathfrak{p}}\left(\mathbb{F}_{q^i}\right)}{i} T^i\right) = \frac{1}{(1-T)P_2(X,T)(1-q^2T)}$$

where

$$P_2(X,T) := \det\left(1 - T\,\mathrm{Frob}_{\mathfrak{p}}\,|H^2\right) \in \mathbb{Z}[t]$$

which have reciprocal roots of absolute value q.

Theorem (Tate Conjecture) [Tate], [Charles], [Pera] and [Maulik]

X an abelian surface *or* a K3 surface then:

- $\rho(X_{\mathfrak{p}}) = \mathrm{ord}_{T=1/q} P_2(T)$
- $\rho(\overline{X}_{\mathfrak{p}}) = \sum_{\zeta} \mathrm{ord}_{T=\zeta/q} P_2(T)$, where ζ runs over all roots of unity.
- (Artin-Tate Conjecture) $P_2(T) \rightsquigarrow \mathrm{disc}(\mathrm{NS}(X_{\mathfrak{p}})) \bmod \mathbb{Q}^{\times 2}$

Partial answer

Question

How are the geometric Picard numbers, $\rho(\overline{X})$ and $\rho(\overline{X}_{\mathfrak{p}})$, related?

Theorem [Charles]

Charles constructed a function $\eta(\overline{X}) \geq 0$[a] and proved that for all $\mathfrak{p}$ of good reduction we have

$$\min_{\mathfrak{q}} \rho(\overline{X}_{\mathfrak{q}}) = \rho(\overline{X}) + \eta(\overline{X}) \leq \rho(\overline{X}_{\mathfrak{p}})$$

- Equality occurs infinitely often.
- Furthermore, over some finite extension of k, the set of such primes has density one.

[a] Depends on the Hodge structure underlying the transcendental lattice and its endomorphism algebra.

Picard Jumps

Let

$$\Pi_{\mathrm{jump}}(X) := \{\mathfrak{p} : \rho(\overline{X}) + \eta(\overline{X}) < \rho(\overline{X}_{\mathfrak{p}})\}$$
$$= \left\{\mathfrak{p} : \min_{\mathfrak{q}} \rho(\overline{X}_{\mathfrak{q}}) < \rho(\overline{X}_{\mathfrak{p}})\right\}$$

Question

What can we say about $\Pi_{\mathrm{jump}}(X)$?
What about

$$\gamma(X, B) := \frac{\#\{\|\mathfrak{p}\| \leq B : \mathfrak{p} \in \Pi_{\mathrm{jump}}(X)\}}{\#\{\|\mathfrak{p}\| \leq B\}} \text{ as } B \to \infty \quad ?$$

Why?

Information about $\Pi_{\mathrm{jump}}(X) \rightsquigarrow$ Geometric statements

Theorem [Bogomolov-Hassett-Tschinkel] and [Li-Liedtke]

If

$$\#\Pi_{\mathrm{jump}}(X) = +\infty \text{ or } \eta(\overline{X}) > 0$$

then $\overline{X}$ has infinitely many rational curves.

Product of Elliptic Curves

Let $X \simeq \text{Kummer}(E_1 \times E_2)$ and E_i elliptic curve over $\mathbb{Q}$.

$$\rho(X) = 18 + \text{rk}(\text{Hom}(E_1, E_2)) = 18 + \begin{cases} 0 & E_1 \not\sim E_2; \\ \text{rk}(\text{End}(E_1)) & E_1 \sim E_2 \end{cases}$$

X	$\rho\left(\overline{X}\right)$	$\gamma(X, B)$	
square of CM	20	$1/2 + o(1)$	CM theory
square of non-CM	19	$\frac{\log\log B \log B}{B} < \bullet < C\frac{\log B}{B^{1/4}}$	[Elkies]
CM times CM	18	$1/4 + o(1)$	CM theory
CM times non-CM	18	?	
non-CM times non-CM	18	$\gg 0$	[Charles]

Remark

$p \in \Pi_{\text{jump}}(X)$ depends uniquely on the pair $(a_{E_1}(p), a_{E_2}(p))$

The simplest case

Let $X \simeq \text{Kummer}(E_1 \times E_2)$ and E_i elliptic curve over $\mathbb{Q}$.

$$\rho(X) = 18 + \text{rk}(\text{Hom}(E_1, E_2)) = 18 + \begin{cases} 0 & E_1 \not\sim E_2; \\ \text{rk}(\text{End}(E_1)) & E_1 \sim E_2 \end{cases}$$

X	$\rho\left(\overline{X}\right)$	$P\left(p \in \Pi_{\text{jump}}(X)\right)$	$\gamma(X, B)$
square of CM	20	$1/2$	$1/2$
square of non-CM	19	$\sim 1/\sqrt{p}$	$c/\sqrt{B}$
CM times CM	18	$1/4$	$1/4$
CM times non-CM	18	$\sim 1/\sqrt{p}$	$c/\sqrt{B}$
non-CM times non-CM	18	$\sim 1/\sqrt{p}$	$c/\sqrt{B}$

Remark

$p \in \Pi_{\text{jump}}(X)$ depends uniquely on the pair $(a_{E_1}(p), a_{E_2}(p))$

Sato-Tate for Abelian Surfaces

Let A be an abelian surface.

Let $G \subset \mathrm{Sp}_4$ be the "smallest" group such that for each $\mathfrak{p}$ of good reduction there is an $g \in G$ such that

$$\det(1 - Tg) = \det(1 - T/\sqrt{q}\,\mathrm{Frob}_{\mathfrak{p}}\,|H^1(A)).$$

Then $\mathrm{ST}_A \subset \mathrm{USp}_4$ is the maximal compact subgroup of G.

Conjecture [Sato-Tate]

The conjugacy classes of $\mathrm{Frob}_{\mathfrak{p}}/\sqrt{q}|H^1$ are equidistributed with respect to the Haar measure on ST_A.

Theorem [Fité–Kedlaya–Rotger–Sutherland]

The Galois structure on $\mathrm{End}(\overline{A}) \otimes \mathbb{R}$ determines ST_A.
Only 52 groups up to conjugacy can be realized as ST_A

Example, if $\mathrm{End}(\overline{A}) = \mathbb{Z}$, then $\mathrm{ST}_A = \mathrm{USp}_4$.

Sato-Tate for Abelian Surfaces

Conjecture [Sato-Tate]

The conjugacy classes of $\mathrm{Frob}_{\mathfrak{p}} / \sqrt{q} | H^1$ are equidistributed with respect to the Haar measure on ST_A.

If $\det\left(1 - T/\sqrt{q}\,\mathrm{Frob}_{\mathfrak{p}}\,|H^1\right) := 1 + a_1 T + a_2 T^2 + a_1 T^3 + T^4$, then

$$(a_1, a_2) \sim (\mathrm{Tr}(M), \mathrm{Tr}(M \wedge M)) \text{ with } M \in \mathrm{ST}_A.$$

Question

What about $\det\left(1 - T/q\,\mathrm{Frob}_{\mathfrak{p}}\,|H^2\right)$?

$H^2 = H^1 \wedge H^1$, thus

$$\det\left(1 - T/q\,\mathrm{Frob}_{\mathfrak{p}}\,|H^2\right) = (T-1)^2 \psi_{\mathfrak{p}}(T)$$
$$\psi_{\mathfrak{p}}(T) := 1 + (2 - a_2)T + (2 + a_1^2 - 2a_2)T^2 + (2 - a_2)T^3 + T^4$$

Abelian Surfaces

Let $X \simeq \text{Kummer}(A)$, A an abelian surface over $\mathbb{Q}$.

$\text{End}(\overline{A}) \rightsquigarrow \rho(\overline{A})$ and ST_A

$\rho(\overline{X}) = 16 + \rho(\overline{A})$

$\overline{A}$	$\rho\left(\overline{A}\right)$	$\gamma(A, B)$ as $B \to \infty$
the generic case	1	$c/\sqrt{B}$
non-CM times non-CM (non Galois)	2	$c/\sqrt{B}$
non-CM times non-CM (Galois) *or* simple RM	2	$1/2$
CM times non-CM	2	$c/\sqrt{B}$
simple CM	2	$3/4$
CM times CM	2	$1/4$
simple QM or square of non-CM	3	$c/\sqrt{B}$
square of CM	4	$1/2$

Remark

Easy to verify the conjectures numerically!

What about K3 surfaces?

Let X be a K3 surface.

In this case $H^1(X)$ is trivial and $\dim H^2(X) = 22$.

In general, we need to compute $\det(1 - T/q\,\mathrm{Frob}_{\mathfrak{p}}\,|H^2(X))$ to deduce $\rho(\overline{X}_{\mathfrak{p}})$!

The definition of ST_X follows closely the one for an abelian surface. We just need to replace

$$\det(1 - T/\sqrt{q}\,\mathrm{Frob}_{\mathfrak{p}}\,|H^1(A)) \text{ by } \det(1 - T/q\,\mathrm{Frob}_{\mathfrak{p}}\,|H^2(X)).$$

$$\mathrm{ST}_X \subset \mathrm{O}_{22-\rho(X)} \text{ and } \mathrm{ST}_X^0 \subset \mathrm{SO}_{22-\rho(\overline{X})}.$$

Very little is known about what subgroups of O_{21} can show up as ST_X!

Kummer surface

If $X \simeq \text{Kummer}(A)$, where A is an abelian surface, then

$$\rho(\overline{X}) = 16 + \rho(\overline{A})$$

and

$$\mathrm{ST}_X = \mathrm{ST}_A/\{\pm 1\} \subset \mathrm{SO}_5 \cong \mathrm{USp}_4/\{\pm 1\}.$$

Computing $\rho(\overline{X})$

Let T_X be the orthogonal complement of $\mathrm{NS}(X_{\mathbb{C}}^{\mathrm{top}})$ in $H^2(X_{\mathbb{C}}^{\mathrm{top}}, \mathbb{Q})$.
Let E_X be the endomorphism algebra of T_X that respects the Hodge structure
E_X is a totally real field or a CM-field.

Theorem [Charles]

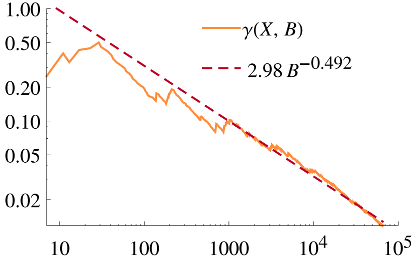
$$\rho(\overline{X}_{\mathfrak{p}}) \geq \begin{cases} \rho(\overline{X}) & \text{if } E_X \text{ is a CM or } \dim_{E_X}(T_X) \text{ is even,} \\ \rho(\overline{X}) + [E_X : \mathbb{Q}] & \text{if } E_X \text{ is a totally real or } \dim_{E_X}(T_X) \text{ is odd.} \end{cases}$$

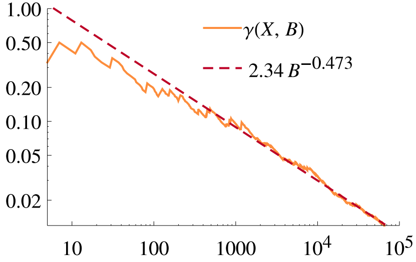
Further, assume that we are in the second case, then exist infinitely many pairs $(\mathfrak{p}, \mathfrak{q})$ such that the equality holds and

$$\mathrm{disc}(\mathrm{NS}(X_{\mathfrak{p}})) \not\equiv \mathrm{disc}(\mathrm{NS}(X_{\mathfrak{q}})) \bmod \mathbb{Q}^{\times 2}$$

Numerical experiments for $\rho(\overline{X}) = 1$

X a quartic K3 surface with $\rho(\overline{X}) = 1$ and $E_X = \mathbb{Q}$.





$$\gamma(X, B) \sim c_X/\sqrt{B}, \quad B \to \infty$$

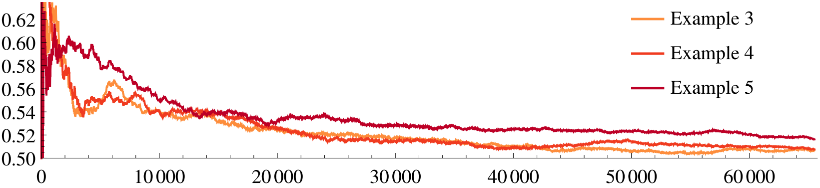
$$\mathrm{Prob}(\mathfrak{p} \in \Pi_{\mathrm{jump}}(X)) \sim 1/\sqrt{p}$$

Heuristics for the $1/\sqrt{p}$ behaviour?!?

k	1	2	3	4	5
$\mathbb{E}\left[\mathrm{Tr}(\mathrm{Frob}_{\mathfrak{p}}/p)^k\right]$	0.002	0.988	-0.016	2.866	-0.218
	-0.008	1.016	-0.065	3.096	-0.658
$\mathbb{E}_{\mathrm{O}}[Tr(M)^k]$	0	1	0	3	0

Numerical experiments for $\rho(\overline{X}) = 2$

Now $\rho(X) = \rho(\overline{X}) = 2$ and $E_X = \mathbb{Q}$ or CM



No obvious trend ...

Is it related to the splitting of primes in a quadratic extension over $\mathbb{Q}$?
Hint: $\mathbb{Q}(\sqrt{D_X})$.

k	1	2	3	4	5
	0.022	1.009	0.0573	2.959	0.136
$\mathbb{E}\left[\mathrm{Tr}(\mathrm{Frob}_\mathfrak{p}/p)^k\right]$	0.001	0.990	-0.039	2.941	-0.454
	0.009	1.013	0.055	3.069	0.015
$\mathbb{E}_\mathrm{O}[Tr(M)^k]$	0	1	0	3	0

$\sim$9000 CPU hours per example.

Discriminant of a K3 surface

Let X be a quartic K3 surface over $\mathbb{Q}$ and D_X be its discriminant.

Theorem [Elsenhans-Jahnel]

The functional equation of the Frobenius action on $H^2(X)$ has the plus sign if and only if D_X is square $\bmod p$.

Theorem

If $\rho(X) = 2r$ then $\rho(\overline{X}_{\mathfrak{p}}) \geq 2r + 2$ at all primes p such that D_X is not square mod p.

Corollary

If D_X is not a square, $\rho(X) = \rho(\overline{X}) = 2r$ and $\eta(\overline{X}) = 0$, then

$$\left\{ p : p \text{ is inert in } \mathbb{Q}\left(\sqrt{D_X}\right) \right\} \subset \Pi_{\mathrm{jump}}(X),$$

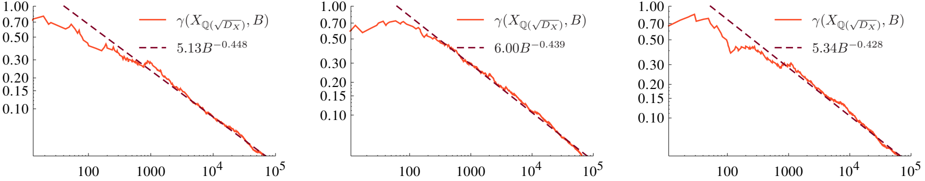
$$\liminf_{B \to \infty} \gamma(X, B) \geq 1/2,$$

and $\overline{X}$ has infinitely many rational curves.

Numerical experiments for $\rho(X) = \rho(\overline{X}) = 2$

If we "ignore" $\left\{ p : p \text{ is inert in } \mathbb{Q}\left(\sqrt{D_X}\right)\right\} \subset \Pi_{\mathrm{jump}}(X)$

$$\gamma\left(X_{\mathbb{Q}(\sqrt{D_X})}, B\right) \sim c/\sqrt{B}, \quad B \to \infty$$



$$\mathrm{Prob}(p \in \Pi_{\mathrm{jump}}(X)) = \begin{cases} 1 & \text{if } p \text{ is inert in } \mathbb{Q}\left(\sqrt{D_X}\right), \\ \sim \frac{1}{\sqrt{p}} & \text{if } p \text{ splits in } \mathbb{Q}\left(\sqrt{D_X}\right) \end{cases}$$

Heuristics for the $1/\sqrt{p}$ behaviour?!?

More examples?

Does $\mathbb{Q}\left(\sqrt{D_X}\right)$ play the same role when $\rho(X) < \rho(\overline{X}) = 2r$? No!

But we have a similar trend!

$$\liminf_{B \to \infty} \gamma(X, B) \geq 1/2.$$

Sometimes we can guess the right quadratic extension K such that

$$\{p : \ p \text{ is inert in } K\} \subset \Pi_{\text{jump}}(X)$$

What about when $\rho(\overline{X})$ is odd?

$$\gamma(X, B) \sim c_X / \sqrt{B}, \quad B \to \infty.$$

Summary and Questions

Computing zeta functions of K3 surfaces via p-adic cohomology $\rightsquigarrow$

- Experimental data for $\Pi_{\mathrm{jump}}(X)$
- Results regarding $\Pi_{\mathrm{jump}}(X)$
- New examples of K3 surfaces with infinitely many rational curves

Questions

- Heuristics for the $1/\sqrt{p}$ behaviour?
  "Equidistribution" of the characteristic polynomials doesn't seem to be enough.
- Can we compute (or statistically guess) E_X?
  Can we distinguish CM vs $\mathbb{Q}$?
- For abelian surfaces A, ST_A is determined by its Galois type, i.e.,

$$\rho_A : \mathsf{Gal}(K/k) \hookrightarrow \mathsf{Aut}_{\mathbb{R}-\mathsf{alg}}(\mathsf{End}(A_K)_{\mathbb{R}}).$$

  Can we say something similar for K3 surfaces? involving E_X?

References

Edgar Costa and Yuri Tschinkel: **Variation of Néron-Severi ranks of reductions of K3 surfaces**
Experimental Mathematics 23 (2014), 475-481.

Edgar Costa: **Effective computations of Hasse–Weil zeta functions**
Ph.D. Thesis

Thank you!