Zeta functions of quartic K3 surfaces over $\mathbb{F}_3$

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Explicit p-adic methods, 20th March 2016

Joint work with: David Harvey and Kiran Kedlaya

Setup

$X \subset \mathbb{P}^3_{\mathbb{F}_q}$:= a quartic K3 surface, a smooth surface defined by

$$f(x_0, \ldots, x_3) = 0, \quad \deg f = 4,$$

Then

$$\zeta_X(t) := exp\left(\sum_{a>0} \frac{\#X(\mathbb{F}_{p^a})t^a}{a}\right) \in \mathbb{Q}(t)$$
$$= \frac{1}{(1-t)(1-qt)(1-q^2t)q^{-1}L(qt)},$$

$$L(t) \in \mathbb{Z}[t], \quad \deg L = 21, \quad L(0) = q$$

all roots on the unit circle.

Goal: Compute $L(t)$ efficiently!

Existing algorithms for "generic" hypersurfaces

With p-adic cohomology:

- Lauder–Wan: $p^{2\dim X+2+o(1)}$
- Abbott–Kedlaya–Roe: $p^{\dim X+1+o(1)}$
- Voight – Sperber: $p^{1+\dim X\cdot(\text{failure to be sparse})+o(1)}$
- Lauder's deformation: $p^{2+o(1)}$.
- Pantratz – Tuitman: $p^{1+o(1)}$
- C. – Harvey – Kedlaya: $p^{1+o(1)}$, $p^{1/2+o(1)}$, or $log^{4+o(1)} p$ on average.

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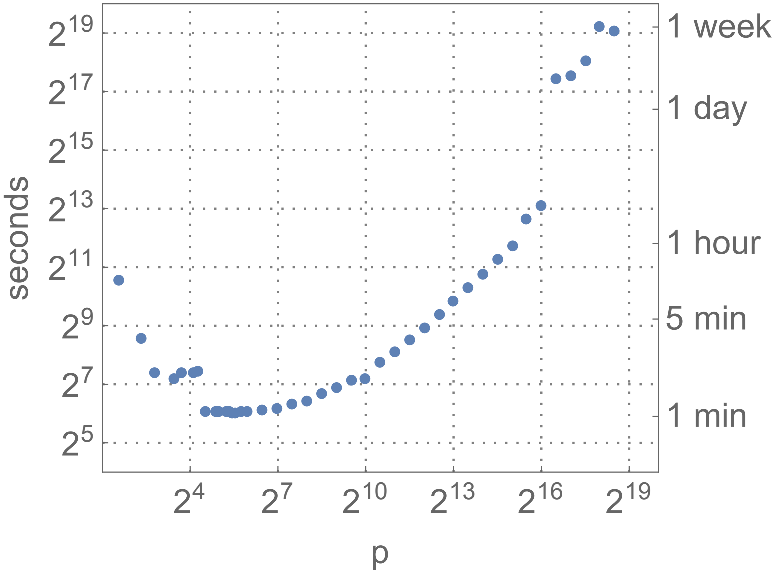
Without "using" p-adic cohomology or smoothness:

- Harvey: $p^{1+o(1)}$, $p^{1/2+o(1)}$, or $log^{4+o(1)}p$ on average.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2^{19}
2^{17}
2^{15}
2^{13}
2^{11}
2^{9}
2^{7}
2^{5}
1 week
1 day
1 hour
5 min
1 min
2^{4}
2^{7}
2^{10}
2^{13}
2^{16}
2^{19}
p

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  Done! [Kedlaya–Sutherland]
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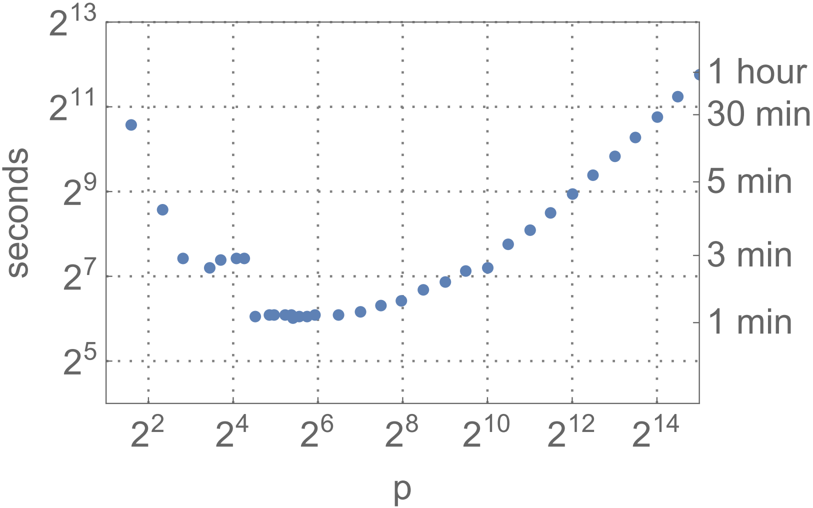
  - C.–Harvey–Kedlaya : almost 25 min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Terms to reduce $=$ $O(p)$ matrix vector multiplications

- $p > 42 \longrightarrow \sim 4000$
- $p = 3 \longrightarrow \sim 130{,}00$

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- No C version yet
  We estimate that should take about 0.5 seconds per surface.