Explicit modularity of K3 surfaces with CM of large degree

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Slides available at `edgarcosta.org`.
Joint work with Andreas-Stephan Elsenhans, Jörg Jahnel, John Voight

From Elliptic Curves to K3 Surfaces: an Analogy 🗺️

Feature	Elliptic curve $E/\mathbb{Q}$	K3 surface $X/\mathbb{Q}$
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Today: three examples with $\dim \mathrm{T}(X) = 6$ and $\mathrm{End}_{\mathrm{Hdg}}(\mathrm{T}(X))_{\mathbb{Q}}$ is a sextic CM field.

The Protagonists: Three degree 2 K3 surfaces with suspicious endomorphisms 🔎

- Three double covers of $\mathbb{P}^2$

$$X_1 \colon w^2 = x\,y\,z\,(x^3 - 3xy^2 + y^3 - 3x^2z - 3xyz + 9y^2z + 6yz^2 + z^3)$$

$$X_2 \colon w^2 = x\,y\,z\,(7x^3 - 7x^2y + y^3 + 49x^2z - 21xyz - 7y^2z + 98xz^2 + 49z^3)$$

$$X_3 \colon w^2 = x\,y\,z\,\left(\begin{array}{c} 49x^3 - 304x^2y + 361xy^2 + 361y^3 + 570x^2z - 2793xyz \\ +2888y^2z + 2033xz^2 - 5415yz^2 + 2299z^3 \end{array}\right)$$

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- The right hand side factors as the product of six general lines in $\mathbb{P}^2$.
- Blowing up the 15 double points yields a smooth K3 with

$$\operatorname{rk}\operatorname{NS} X_i = 1 + 15, \qquad \dim \mathrm{T}(X_i) = 22 - 16 = 6.$$

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- Period matrix of $X_i \rightsquigarrow$ strong evidence of CM by a cyclic sextic field K_i.

  $\mathbb{Q}(\zeta_9 + \zeta_9^{-1}, \sqrt{-1}) \simeq$6.0.419904.1 $\quad$ $\mathbb{Q}(\zeta_7 + \zeta_7^{-1}, \sqrt{-1}) \simeq$ 6.0.153664.1 $\quad$ 6.0.59105344.1
- The lines are defined over the unique (real) cubic subfield.

The Accomplices: Four Hyperelliptic Curves with CM 👀

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Yes, we can!

Theorem (Weng + C-Mascot-Sijsling-Voight)

We have $\mathrm{End}(\mathrm{Jac}(C_i))_{\mathbb{Q}} \simeq K_i$, where

i	$\mathbb{Q}(\sqrt{-1}) \subset K_i$	Defining equation for C_i
1	$\mathbb{Q}(\zeta_9 + \zeta_9^{-1}, \sqrt{-1}) \simeq 6.0.419904.1$	$y^2 = x^7 + 6x^5 + 9x^3 + x$
2	$\mathbb{Q}(\zeta_7 + \zeta_7^{-1}, \sqrt{-1}) \simeq 6.0.153664.1$	$y^2 = x^7 + 7x^5 + 14x^3 + 7x$
3	6.0.8340544.1	$y^2 = x^7 + 1786x^5 + 44441x^3 + 278179x$
4	6.0.59105344.1	$y^2 = x^7 + 961x^5 - 3694084x^3 + 1832265664x$

Main Theorem: Matching Modularity

Theorem (C–Elsenhans–Jahnel–Voight)

For $i = 1, 2, 3$, let $X = X_i$ and let K be the predicted sextic CM field. and $F \subseteq K$ the unique cubic subfield.

If $\mathrm{T}_{\mathbb{Q}}(X)$ has CM by K, for an explicit character ψ_X with ∞–type $\{(0,2),(1,1),(1,1)\}$ and for all primes ℓ we have

$$\rho_{\mathrm{T}(X),\ell} \simeq \mathrm{Ind}_{\mathrm{Gal}_K}^{\mathrm{Gal}_{\mathbb{Q}}} \psi_X \qquad L(\mathrm{T}(X), s) = L(s, \psi_X).$$

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- Let $A = \mathrm{Jac}(C_i)$. For an explicit character ψ_A of ∞–type $\{(0,1),(0,1),(0,1)\}$ and for all primes ℓ we have

$$\rho_{\mathrm{H}^1(A),\ell} \simeq \mathrm{Ind}_{\mathrm{Gal}_K}^{\mathrm{Gal}_{\mathbb{Q}}} \qquad L(\mathrm{H}^1(A), s) = L(s, \psi_A).$$

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💥 We have

$$\rho_{\mathrm{H}^2(A),\ell} \simeq \mathrm{Ind}_{\mathrm{Gal}_F}^{\mathrm{Gal}_{\mathbb{Q}}} \mathbb{Q}_\ell(1) \oplus \mathrm{Ind}_{\mathrm{Gal}_K}^{\mathrm{Gal}_{\mathbb{Q}}}(\psi_X \oplus \psi')$$

$$L(\mathrm{H}^2(A), s) = \zeta_F(s+1) L(s, \psi_X) L(s, \psi'),$$

where ψ' is of ∞-type $\{(0,2),(0,2),(1,1)\}$.

How We Pin Down ψ_X: A Four Step Detective Story 🕵️

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5. This example took about 180h (all the others took less than a minute).

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