Distributions in arithmetic geometry

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Presented at Communicating Mathematics Effectively

Slides available at `edgarcosta.org` under Research

Elliptic curves

$$E : y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Z}$$

Write $E_p := E \bmod p$

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$\rightsquigarrow$ studying the **statistical** properties $\#E_p$.

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In other words,

$$\lambda_p := \frac{p + 1 - \#E_p}{\sqrt{p}} \in [-2, 2]$$

What can we say about the error term, λ_p, as $p \to \infty$?

Two types of elliptic curves

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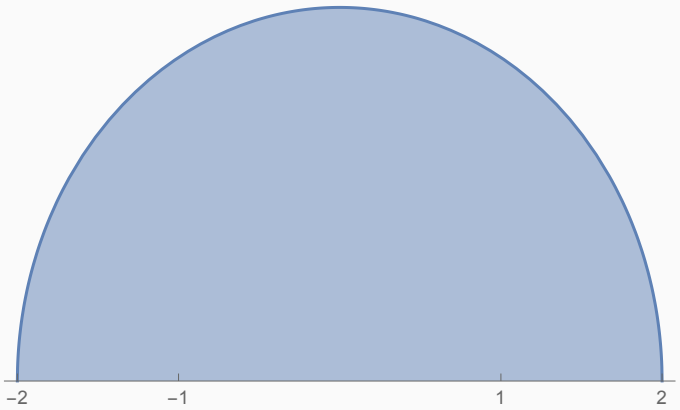

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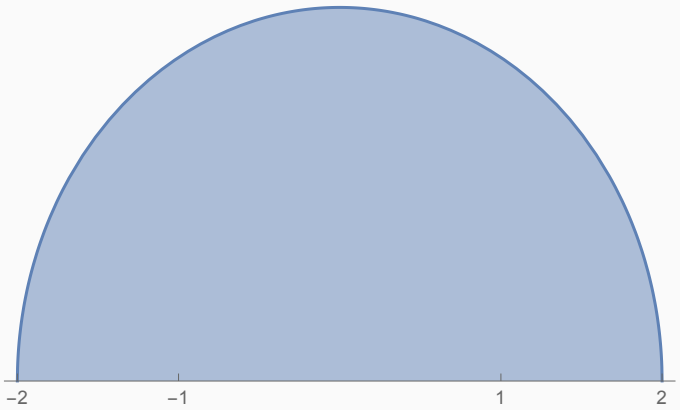

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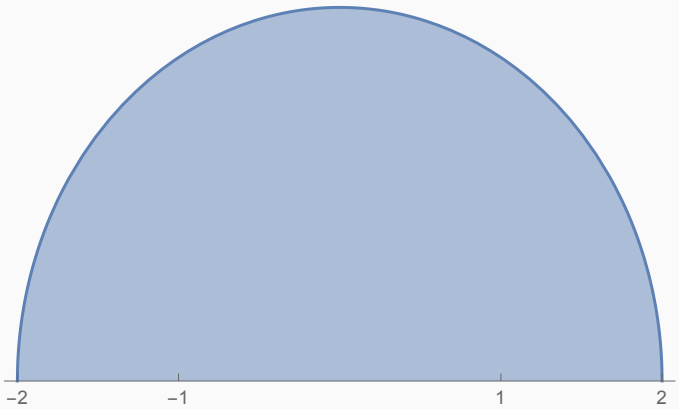

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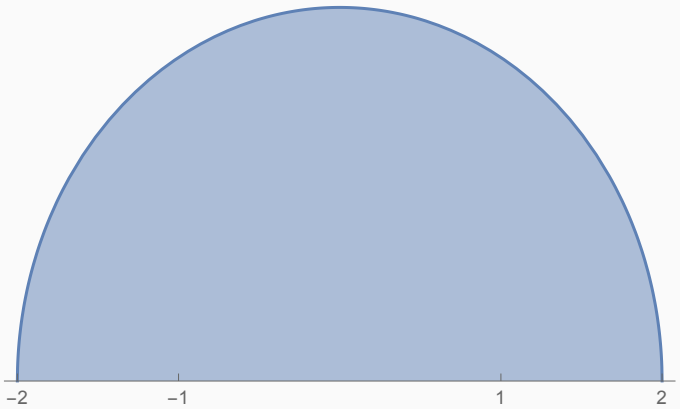

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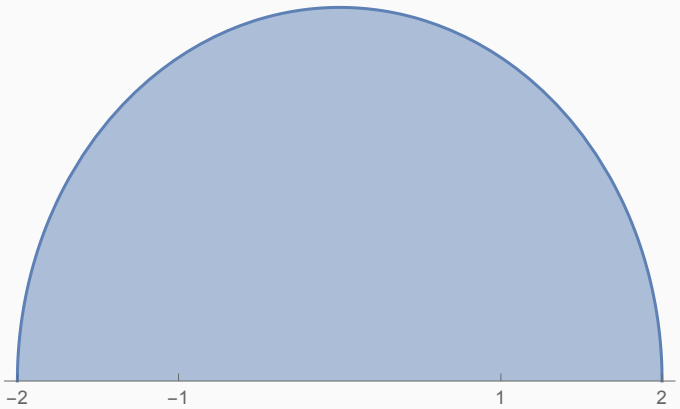

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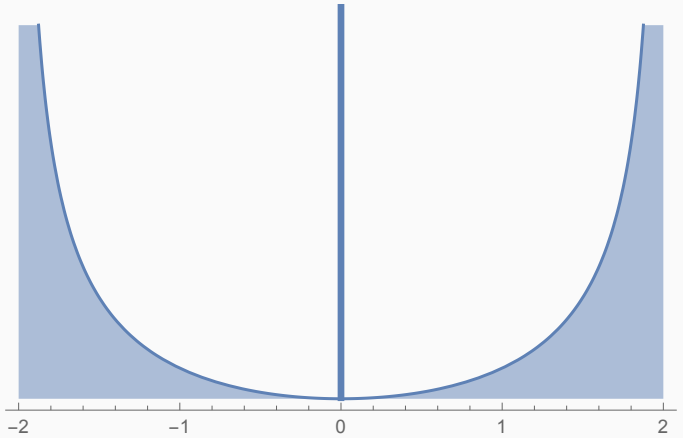

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$$\lambda_p = 0 \iff \mathrm{rk}\,\mathrm{End}\, {E_p}^{\mathrm{al}} > 2 = \min_q \mathrm{rk}\,\mathrm{End}\, E_q^{\mathrm{al}}$$

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This is analogous to studying $\operatorname{rk} \mathrm{End}\, {E_p}^{\mathrm{al}} = \operatorname{rk} \mathrm{NS}({E_p}^{\mathrm{al}} \times {E_p}^{\mathrm{al}})$

Néron–Severi group

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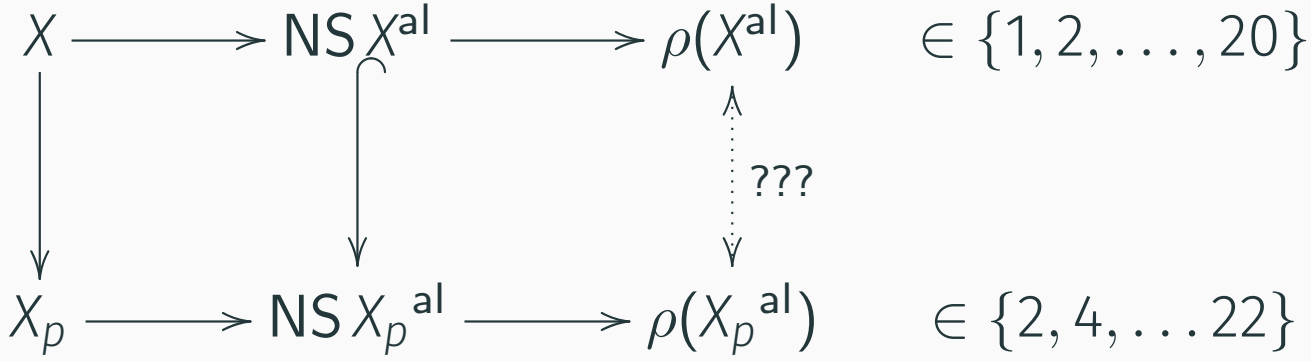

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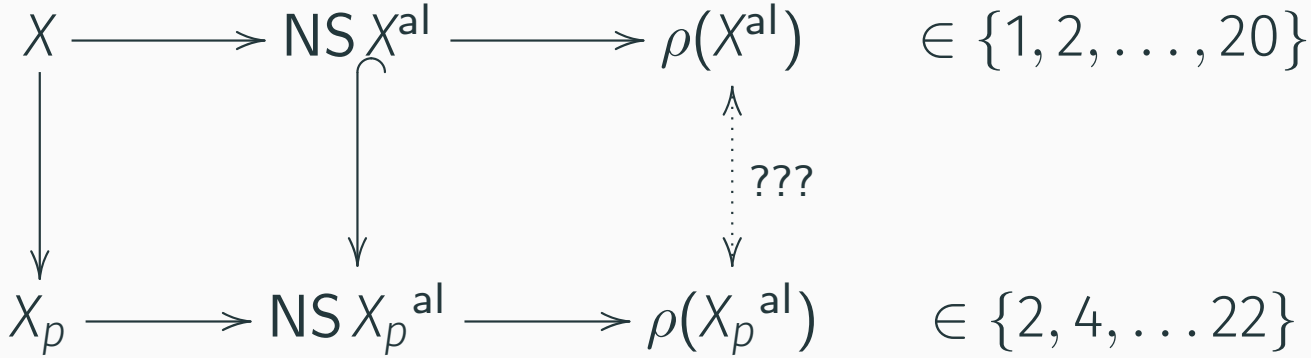
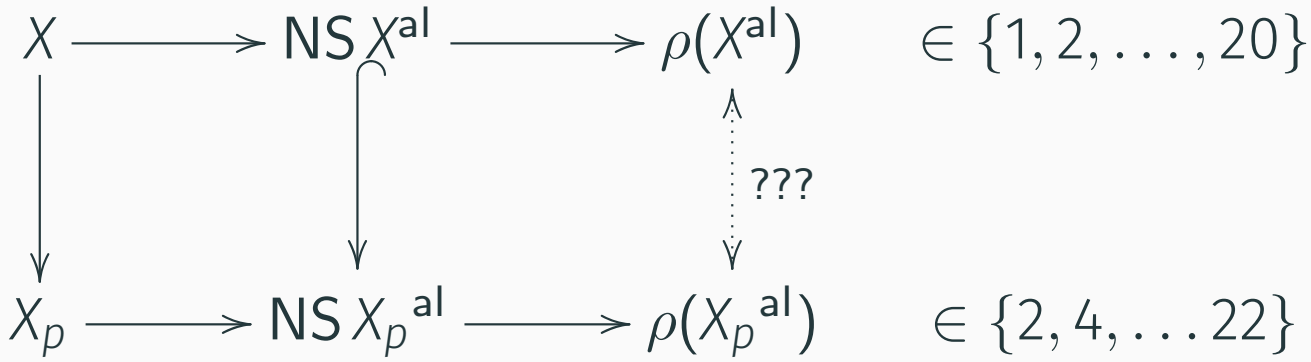


Theorem (Charles)

For infinitely many p we have $\rho(X_p{}^{\mathsf{al}}) = \min_q \rho(X_q{}^{\mathsf{al}})$.

The Problem



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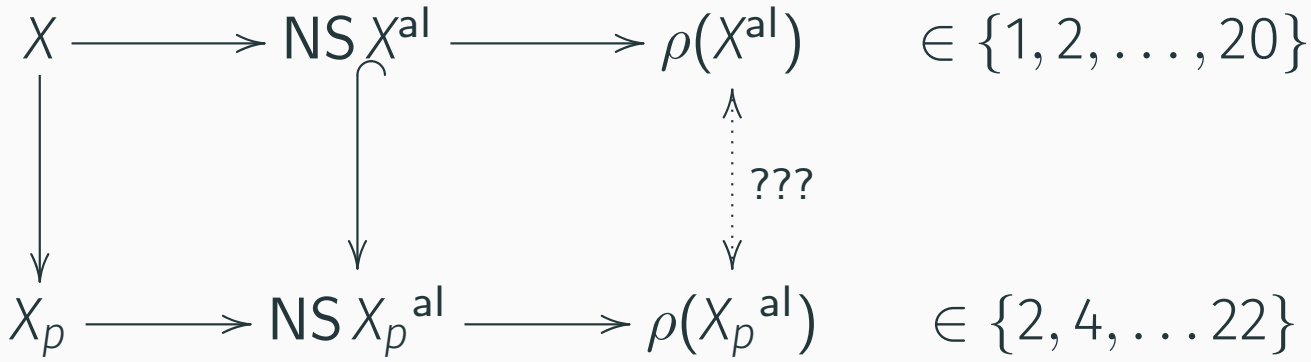

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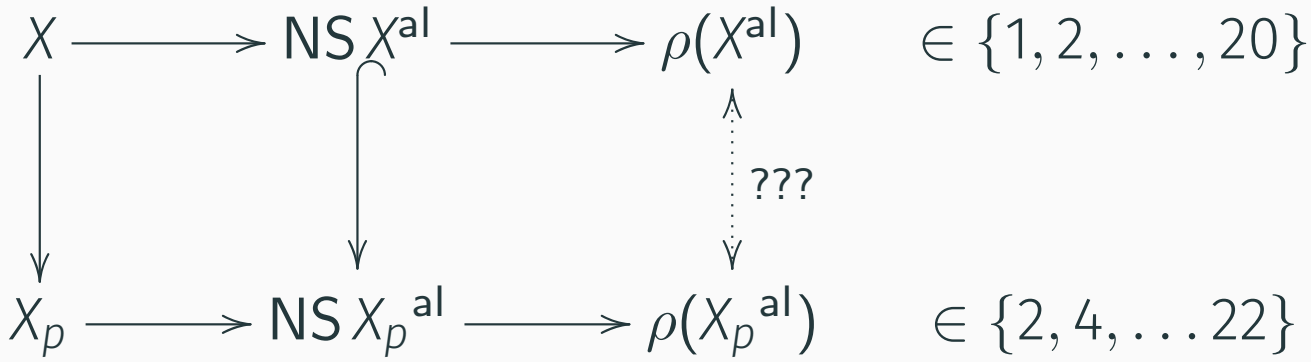

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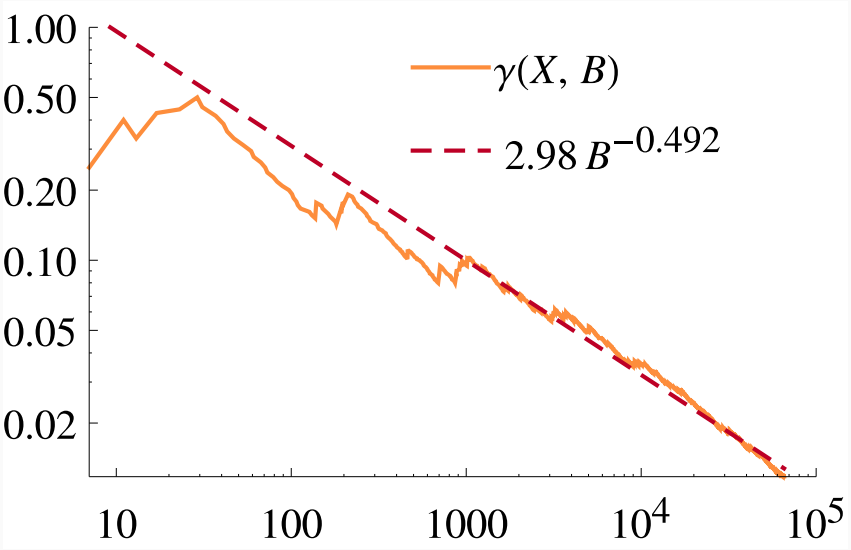
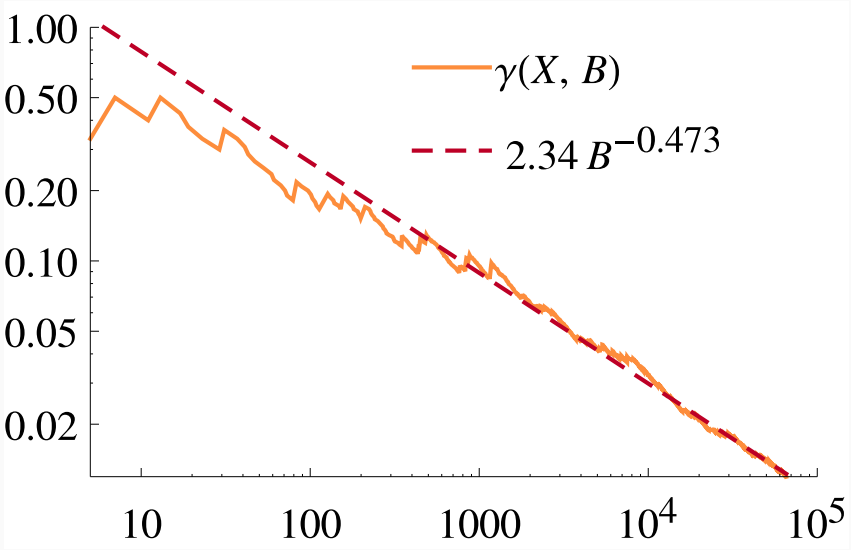
Let's do some numerical experiments!

Generic K3 surfaces, $\rho(X^{\mathrm{al}}) = 1$





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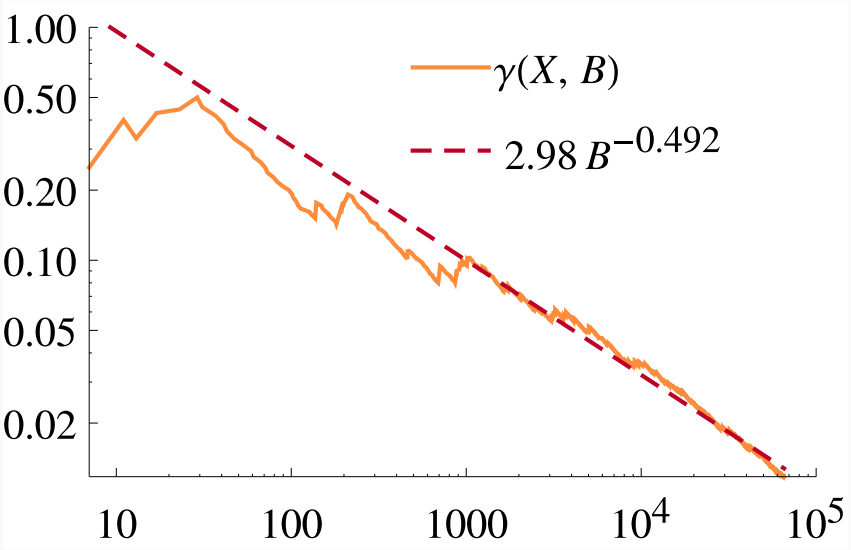

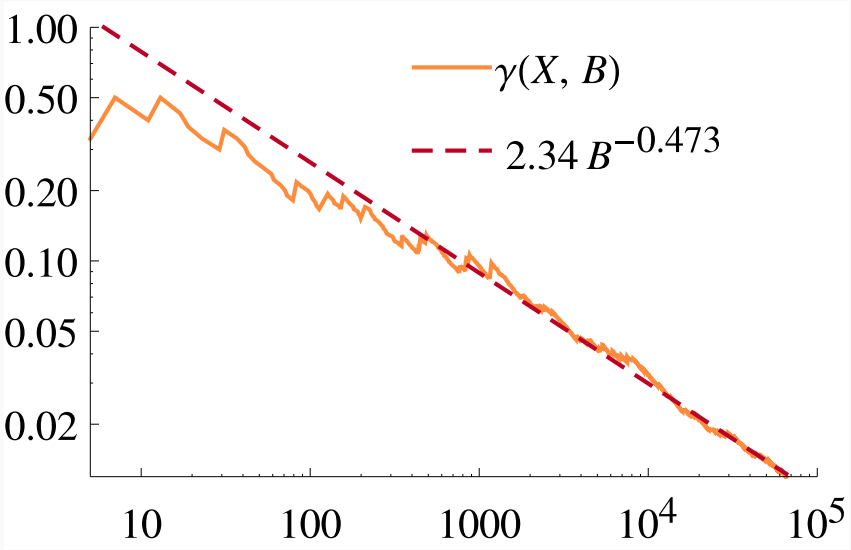

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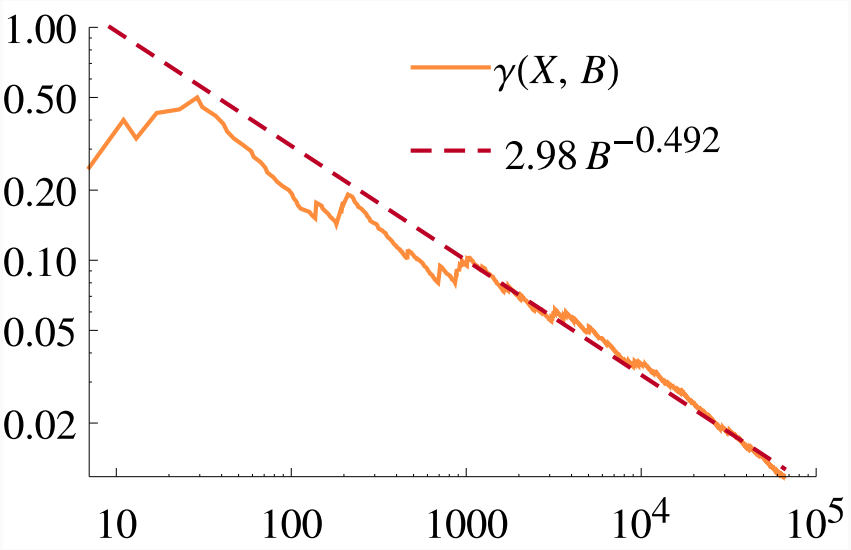

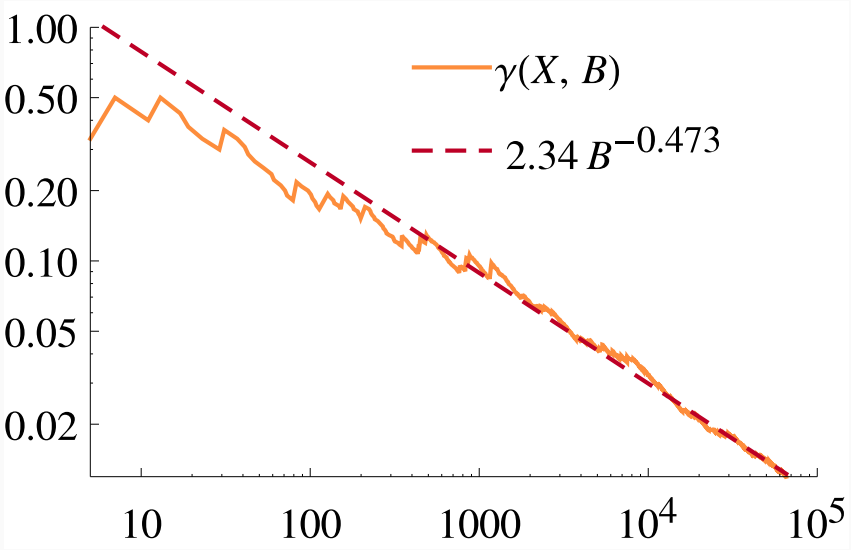

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Why?

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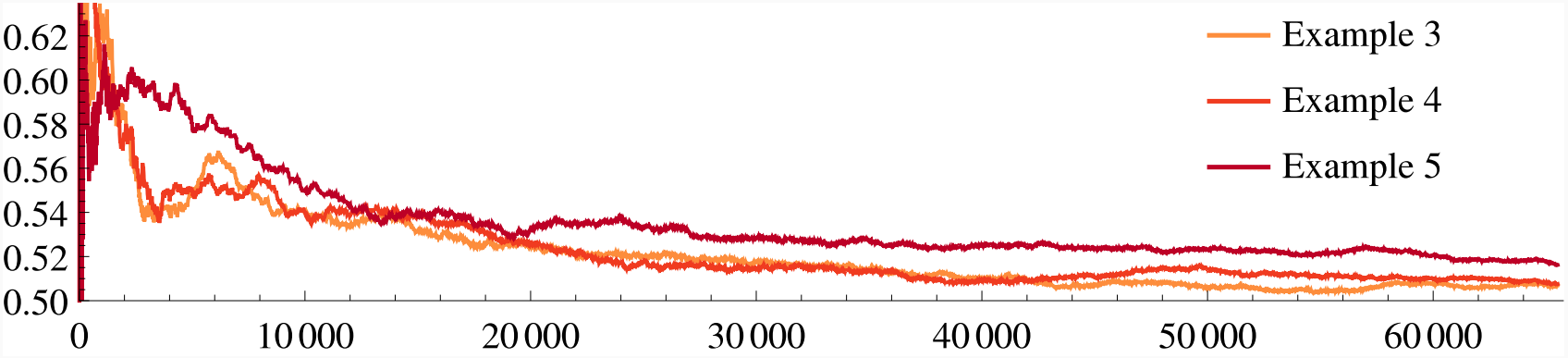

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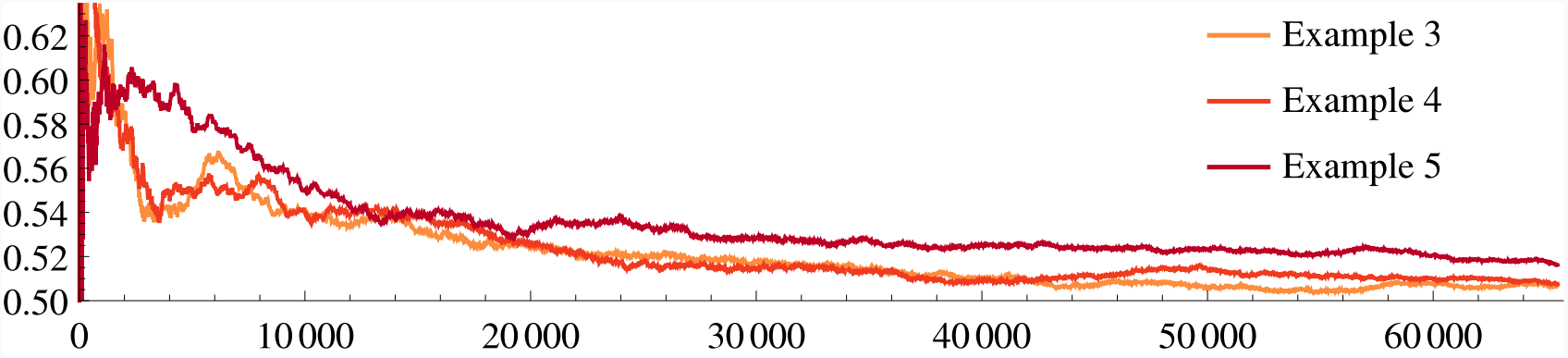

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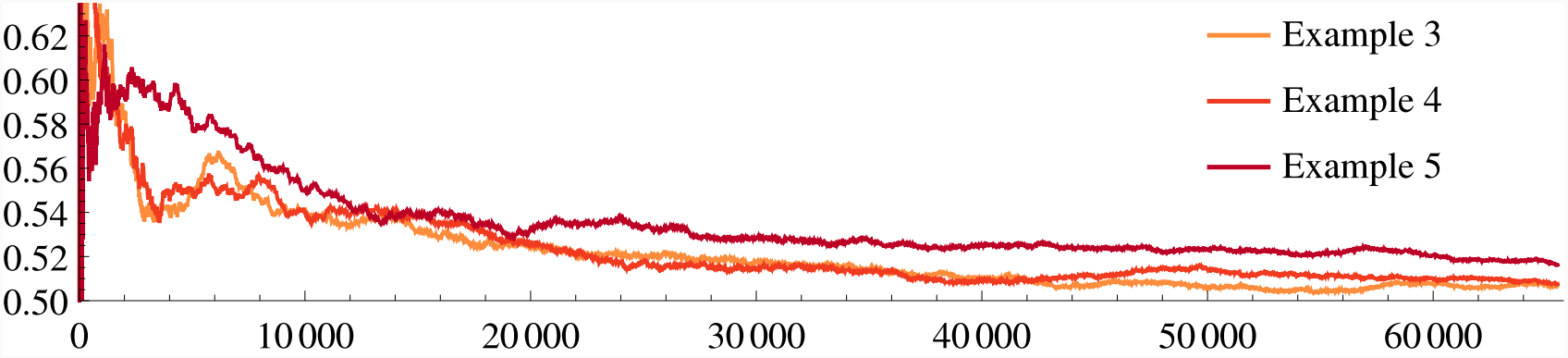

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Could it be related to some integer being a square modulo p?

Numerical experiments $\rightsquigarrow$ Theoretical Results

In most cases we can explain the 1/2!

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Theorem (C, C–Elsenhans–Jahnel)

If $\rho(X^{\mathrm{al}}) = \min_q \rho({X_p}^{\mathrm{al}})$, *then there is a* $d_X \in \mathbb{Z}$ *such that:*

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Corollary

If d_X *is not a square:*

- $\liminf_{B\to\infty} \gamma(X, B) \geq 1/2$
- X^{al} *has infinitely many rational curves.*

Experimental data for $\rho(X^{\mathrm{al}}) = 2$ (again)

What if we ignore $\left\{p > 2 : p \text{ inert in } \mathbb{Q}(\sqrt{d_X})\right\} \subset \Pi_{\mathrm{jump}}(X)$?

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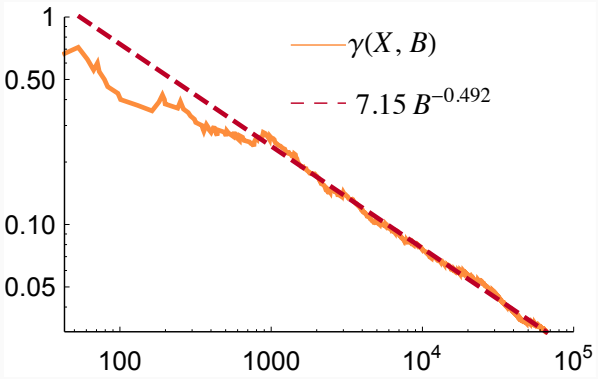

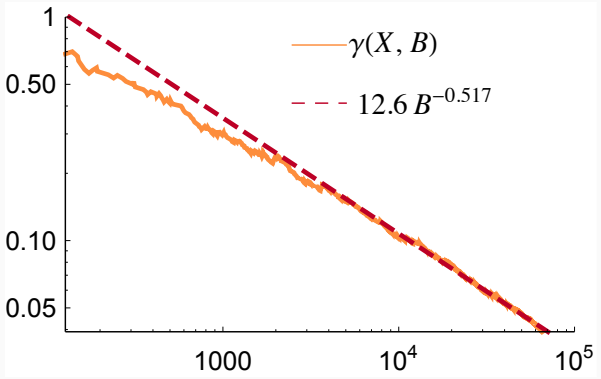

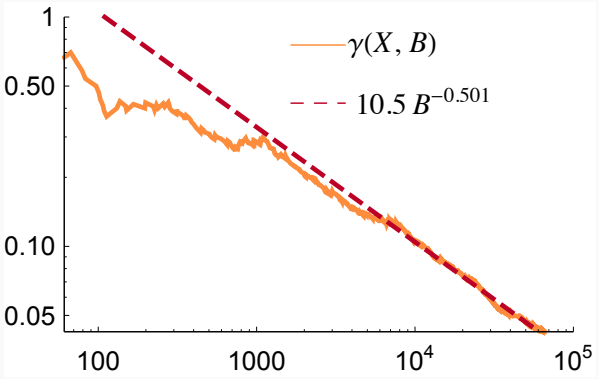

$$\mathrm{Prob}(p \in \Pi_{\mathrm{jump}}(X)) = \begin{cases} 1 & \text{if } d_X \text{ is not a square modulo } p \\ \sim \frac{1}{\sqrt{p}} & \textit{otherwise} \end{cases}$$

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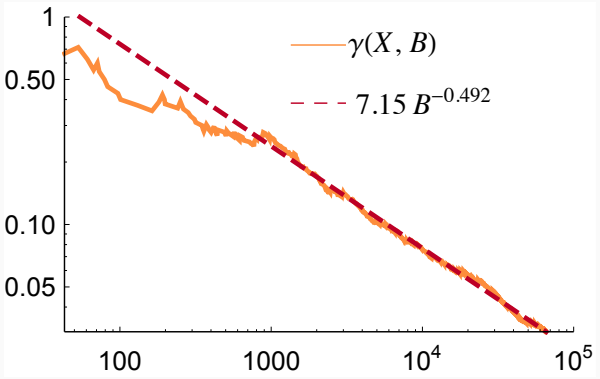

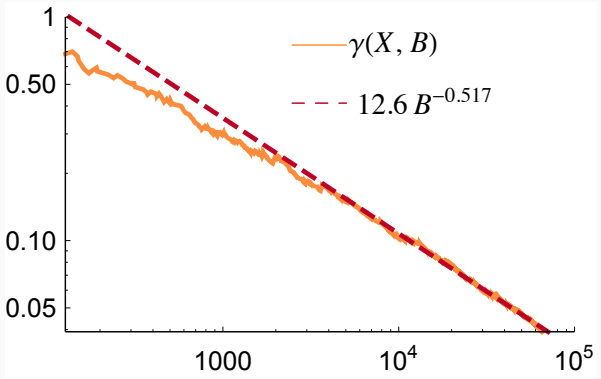

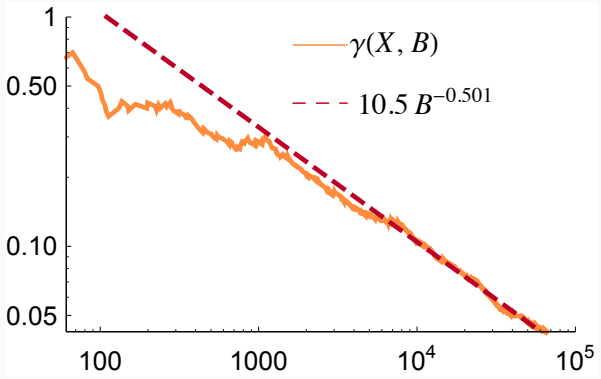

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