Equidistributions in arithmetic geometry

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Motivation: Randomness Principle

Rigidity/Randomness Dichotomy [Sarnak]

Given an arithmetic problem, either

1. rigid structure $\rightsquigarrow$ rigid solution, or
2. the answer is difficult to determine $\rightsquigarrow$ random behaviour

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- Understanding and providing the probability law $\rightsquigarrow$ deep understanding of the phenomenon
- Real world applications

Motivation: Problem

p a prime number
X an "integral" object, e.g.:

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- a polynomial with integer coefficients
- a curve or a surface defined by a polynomial equation with integer coefficients
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Given X mod p

- What can we say about X?
- What if we consider infinitely many primes?
- How does it behave as $p \to \infty$?

Overview

1. Polynomials in one variable
2. Elliptic curves
3. Quartic surfaces

Counting roots of polynomials

$f(x) \in \mathbb{Z}[x]$ an irreducible polynomial of degree $d > 0$

p a prime number

Consider:

$$\begin{aligned} N_f(p) &:= \#\left\{x \in \{0, \dots, p-1\} \;:\; f(x) \equiv 0 \bmod p\right\} \\ &= \#\left\{x \in \mathbb{F}_p \;:\; f(x) = 0\right\} \end{aligned}$$

$$N_f(p) \in \{0, 1, \dots, d\}$$

Question

How often does each value occur?

Example: quadratic polynomials

$f(x) = ax^2 + bx + c$, $\Delta = b^2 - 4ac$, the discriminant of f.

$$\text{Quadratic formula} \implies N_f(p) = \begin{cases} 0 & \text{if } \Delta \text{ is not a square modulo } p \\ 1 & \Delta \equiv 0 \bmod p \\ 2 & \text{if } \Delta \text{ is a square modulo } p \end{cases}$$

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Half of the numbers modulo p are squares.
Hence, if Δ isn't a square, then $\mathrm{Prob}(\Delta \text{ is a square modulo } p) = 1/2$

$$\implies \mathrm{Prob}(N_f(p) = 0) = \mathrm{Prob}(N_f(p) = 2) = \frac{1}{2}$$

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$$\text{For example, if } \Delta = 5 \text{ and } p > 2, \text{ then } N_f(p) = \begin{cases} 0 & \text{if } p \equiv 2, 3 \bmod 5 \\ 1 & \text{if } p = 5 \\ 2 & \text{if } p \equiv 1, 4 \bmod 5 \end{cases}$$

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$$f(x) = x^3 - 2 = \left(x - \sqrt[3]{2}\right)\left(x - \sqrt[3]{2}e^{2\pi i/3}\right)\left(x - \sqrt[3]{2}e^{4\pi i/3}\right)$$

$$\operatorname{Prob}\left(N_f(p) = x\right) = \begin{cases} 1/3 & \text{if } x = 0 \\ 1/2 & \text{if } x = 1 \\ 1/6 & \text{if } x = 3. \end{cases}$$

$$f(x) = x^3 - x^2 - 2x + 1 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$$

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The Chebotarëv density theorem

$f(x) = (x - \alpha_1)\dots(x - \alpha_d)$, $\alpha_i \in \mathbb{C}$
$G := \mathrm{Aut}(\mathbb{Q}(\alpha_1, \dots, \alpha_d)/\mathbb{Q}) = \mathrm{Gal}(f/\mathbb{Q})$
$G \subset S_d$, as it acts on the roots $\alpha_1, \dots, \alpha_d$ by permutations.

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Theorem (Chebotarëv, early 1920s)

For $i = 0, \dots, d$, we have

$$\mathrm{Prob}(N_f(p) = i) = \mathrm{Prob}(g \in G : g \textit{ fixes } i \textit{ roots}),$$

where

$$\mathrm{Prob}(N_f(p) = i) := \lim_{N \to \infty} \frac{\#\{p \textit{ prime}, p \leq N, N_f(p) = i\}}{\#\{p \textit{ prime}, p \leq N\}}.$$

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$$S_3 = \left\{ \begin{array}{c} \text{id}, \\ (1 \leftrightarrow 2), (1 \leftrightarrow 3), (2 \leftrightarrow 3), \\ (1 \to 2 \to 3 \to 1), (1 \to 3 \to 2 \to 1) \end{array} \right\}$$

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Prime powers

We may also define

$$N_f\left(p^e\right) = \#\left\{x \in \mathbb{F}_{p^e} \;:\; f(x) = 0\right\}$$

Theorem (Chebotarëv continued)

$$\mathrm{Prob}\left(N_f\left(p\right) = c_1, N_f\left(p^2\right) = c_2, \cdots\right)$$
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Let $f(x) = x^3 - 2$, then $G = S_3$ and:

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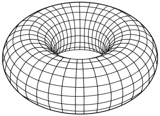
$$\operatorname{Prob}\left(N_f\left(p\right) = 1, N_f\left(p^2\right) = 3\right) = 1/2$$

Elliptic curves

An *elliptic curve* is a smooth plane algebraic curve defined by

$$y^2 = x^3 + ax + b$$

over the complex numbers $\mathbb{C}$ this is a torus: 

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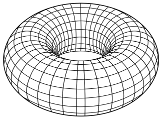
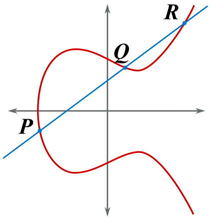


There is a natural *group structure*!
If P, Q, and R are colinear, then
$$P + Q + R = 0$$



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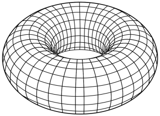
over the complex numbers $\mathbb{C}$ this is a torus: 

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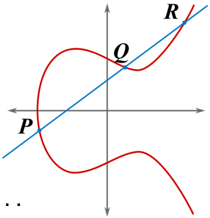
Applications:
- cryptography
- integer factorization
- pseudorandom numbers, ...



Counting points on elliptic curves

Given an elliptic curve over $\mathbb{Q}$

$$X : y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Z}$$

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Instead, we look for **statistical** properties of $N_X(p)$.

Hasse's bound

Theorem (Hasse, 1930s)

$$|p + 1 - N_X(p)| \leq 2\sqrt{p}.$$

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In other words,

$$N_x(p) = p + 1 - \sqrt{p}\lambda_p, \quad \lambda_p \in [-2, 2]$$

What can we say about the error term, λ_p, as $p \to \infty$?

Weil's theorem

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Taking $\lambda_p = 2\cos\theta_p$, with $\theta_p \in [0, \pi]$ we can rewrite

$$N_X(p) = p + 1 - \sqrt{p}(\alpha_p + \overline{\alpha_p}), \quad \alpha_p = e^{i\theta_p}.$$

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Theorem (Weil, 1940s)

$$\begin{aligned} N_X(p^e) &= p^e + 1 - \sqrt{p^e}\left(\alpha_p^e + \overline{\alpha_p}^e\right) \\ &= p^e + 1 - \sqrt{p^e}\, 2\cos(e\,\theta_p) \end{aligned}$$

We may thus focus our attention on

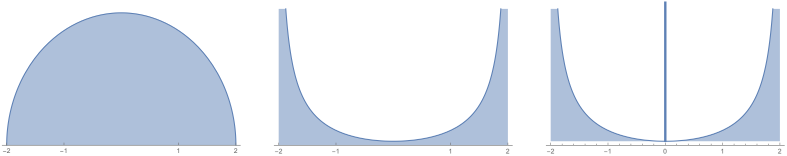
$$p \mapsto \alpha_p \in S^1 \quad \text{or} \quad p \mapsto \theta_p \in [0, \pi] \quad \text{or} \quad p \mapsto 2\cos\theta_p \in [-2, 2]$$

Histograms

If one picks an elliptic curve and computes a histogram for the values

$$\frac{N_X(p) - 1 - p}{\sqrt{p}} = 2\,\mathrm{Re}\,\alpha_p = 2\cos\theta_p$$

over a large range of primes, one always observes convergence to one of three limiting shapes!



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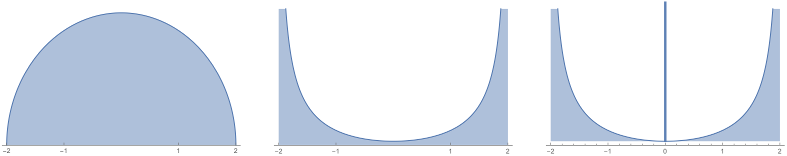

One can confirm the conjectured convergence with high numerical accuracy:
`http://math.mit.edu/~drew/g1SatoTateDistributions.html`

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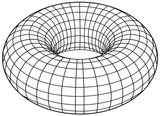
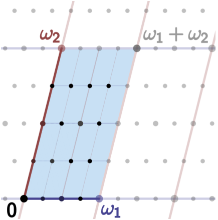
Consider the elliptic curve over $\mathbb{C}$

$X/\mathbb{C} \simeq$  $\simeq \mathbb{C}/\Lambda$ and $\Lambda = \mathbb{Z}\omega_1 \otimes \mathbb{Z}\omega_2 =$ 

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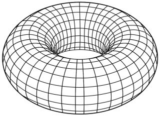
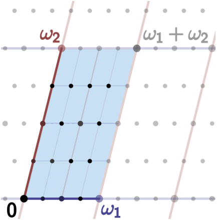
$X/\mathbb{C} \simeq$  $\simeq \mathbb{C}/\Lambda$ and $\Lambda = \mathbb{Z}\omega_1 \otimes \mathbb{Z}\omega_2 =$ 

non-CM the generic case, $\mathrm{End}(\Lambda) = \mathbb{Z}$

CM Λ has extra symmetries,
$\mathbb{Z} \subsetneq \mathrm{End}(\Lambda)$ and $\omega_2/\omega_1 \in \mathbb{Q}(\sqrt{-d})$ for some $d \in \mathbb{N}$.

CM Elliptic curves

Theorem (Deuring 1940s)

If X is a CM elliptic curve then $\alpha_p = e^{i\theta}$ are equidistributed with respect to the uniform measure on the semicircle, i.e.,

$$\left\{ e^{i\theta} \in \mathbb{C} : \operatorname{Im}(z) \geq 0 \right\} \text{ with } \mu = \frac{1}{2\pi}\, \mathrm{d}\theta$$

If the extra endomorphism is not defined over the base field one must take $\mu = \frac{1}{\pi}\, \mathrm{d}\theta + \frac{1}{2}\delta_{\pi/2}$

In both cases, the probability density function for $2\cos\theta$ is

$$\frac{\{1,2\}}{4\pi\sqrt{4-z^2}} =$$

non-CM Elliptic curves

Conjecture (Sato–Tate, early 1960s)

If X does not have CM then $\alpha_p = e^{i\theta}$ are equidistributed in the semi circle with respect to $\mu = \frac{2}{\pi} \sin^2 \theta \, \mathrm{d}\theta$.

The probability density function for $2cos\theta$ is

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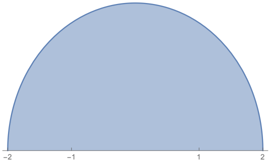

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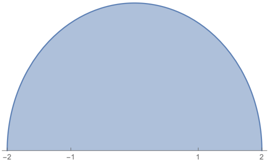

Theorem (Clozel, Harris, Taylor, et al., late 2000s; very hard!)

The Sato–Tate conjecture holds for $K = \mathbb{Q}$ (and more generally for K a totally real number field).

Group-theoretic interpretation

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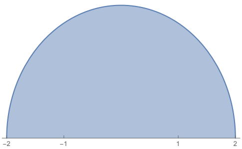
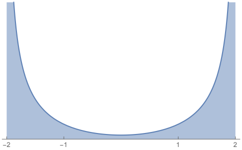
The pairs $\{\alpha_p, \overline{\alpha_p}\}$ are distributed like the eigenvalues of a matrix chosen at random from ST_X with respect to its Haar measure.

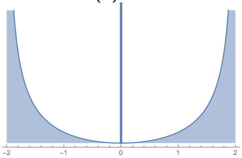
non-CM
$\mathrm{SU}(2)$



CM
$\mathrm{U}(1)$



CM (with the δ)
$\mathrm{N}_{\mathrm{SU}(2)}(\mathrm{U}(1))$



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Instead, we study other geometric invariant.

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$$\begin{tikzcd}
X \arrow[r, dashed] \arrow[d] & \operatorname{Pic} \overline{X} \arrow[r, dashed] \arrow[d, hook] & \rho(\overline{X}) \in [1, \ldots, 20] \arrow[d, dotted, "???"] \\
X_p := X \bmod p \arrow[r, dashed] & \operatorname{Pic} \overline{X}_p \arrow[r, dashed] & \rho(\overline{X}_p) \in [2, 4, \ldots 22]
\end{tikzcd}$$

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X \arrow[r, dashed] \arrow[d] & \operatorname{Pic} \overline{X} \arrow[r, dashed] \arrow[d, hook] & \rho(\overline{X}) \in [1, \ldots, 20] \arrow[d, dotted, "???"] \\
X_p := X \bmod p \arrow[r, dashed] & \operatorname{Pic} \overline{X}_p \arrow[r, dashed] & \rho(\overline{X}_p) \in [2, 4, \ldots 22]
\end{tikzcd}$$

Theorem (Charles 2011)

We have $\min_q \rho(\overline{X}_q) = \rho(\overline{X}_p)$ *for infinitely many* p.

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Information about $\Pi_{\mathrm{jump}}(X) \rightsquigarrow$ Geometric statements

- How often an elliptic curve has $p + 1$ points modulo p?
- How often two elliptic curves have the same number of points modulo p?
- Does $\overline{X}$ have infinitely many rational curves ?
- ...

Numerical experiments for a generic K3, $\rho(\overline{X}) = 1$

$\rho(\overline{X})$ is very hard to compute
$\rho(\overline{X}_p)$ only now computationally feasible for large p [C.-Harvey]

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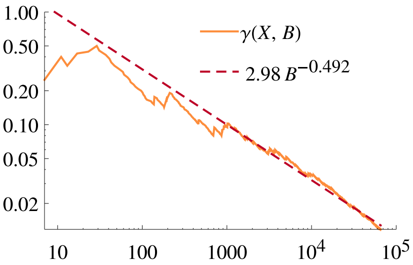

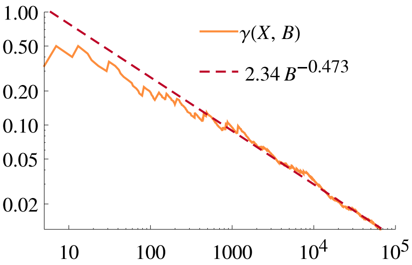

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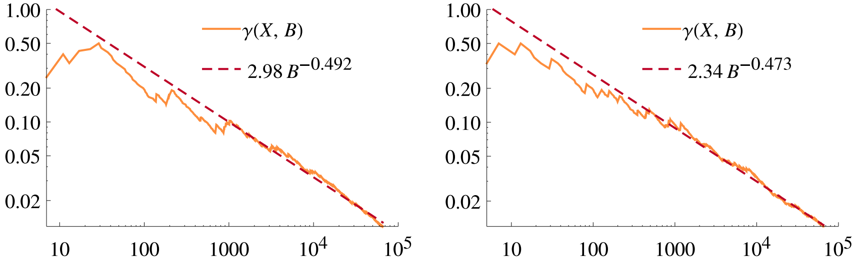

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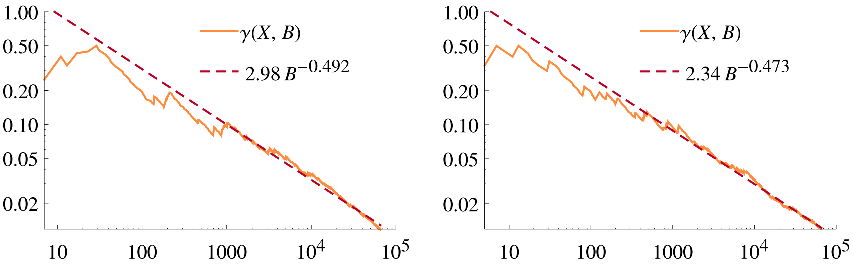

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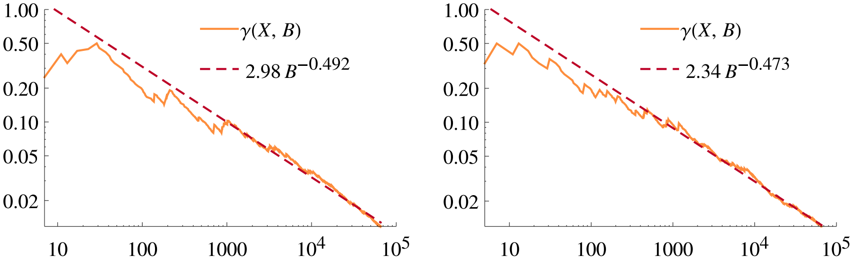

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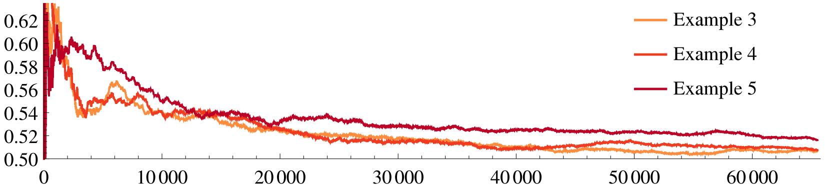
Similar behaviour observed in other examples with $\rho(\overline{X})$ odd.

In this case, data $\rightsquigarrow$ equidistribution in $\mathrm{O}(21)$!

Numerical experiments for $\rho(\overline{X}) = 2$



No obvious trend ...

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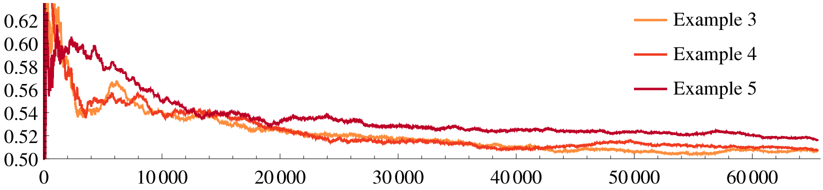

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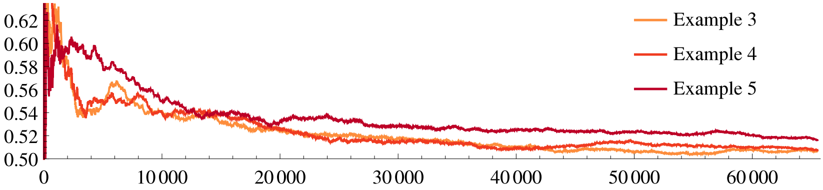

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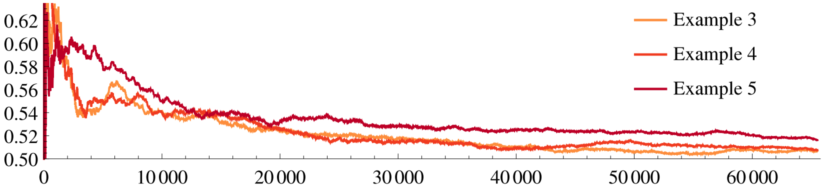

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Data $\rightsquigarrow$ equidistribution in $\mathrm{O}(20)$!

$\sim$1 CPU year per example.

Numerical experiments $\rightsquigarrow$ Theoretical Results

In most cases we can explain the $1/2$!

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Theorem ([C.] and [C.-Elsenhans-Jahnel])

Assume $\rho(\overline{X})$ is even and $\rho(\overline{X}) = \min_q \rho(\overline{X}_q)$, there is a $d_X \in \mathbb{Z}$ such that:

$$\{p > 2 : d_X \text{ is not a square modulo } p\} \subset \Pi_{\mathrm{jump}}(X).$$

In general, d_X is not a square.

Corollary

If d_X is not a square:

- $\liminf_{B\to\infty} \gamma(X, B) \geq 1/2$
- *$\overline{X}$ has infinitely many rational curves.*

Numerical experiments for $\rho(\overline{X}) = 2$, again

What if we ignore $\{p : d_X \text{ is not a square modulo } p\} \subset \Pi_{\mathrm{jump}}(X)$?

Numerical experiments for $\rho(\overline{X}) = 2$, again

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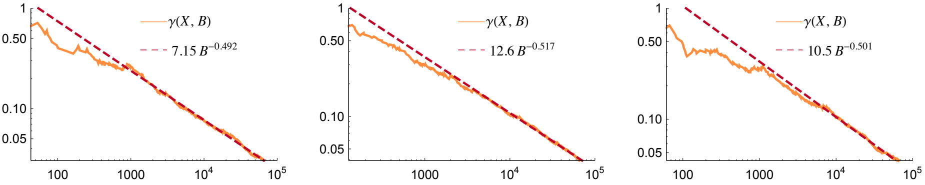
$$\gamma(X, B) \sim c/\sqrt{B}, \quad B \to \infty$$



$$\mathrm{Prob}(p \in \Pi_{\mathrm{jump}}(X)) = \begin{cases} 1 & \text{if } d_X \text{ is not a square modulo } p \\ \sim \frac{1}{\sqrt{p}} & \textit{otherwise} \end{cases}$$

Summary

Computing zeta functions of K3 surfaces via p-adic cohomology$\rightsquigarrow$

- Experimental data for $\Pi_{\mathrm{jump}}(X)$
- Results regarding $\Pi_{\mathrm{jump}}(X)$
- New class of examples of K3 surfaces with infinitely many rational curves

Thank you!