From Frobenius polynomials to geometry

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Simons Collab. on Arithmetic Geometry, Number Theory, and Computation

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Around Frobenius distributions and related topics

Joint work with Andreas-Stephan Elsenhans, Francesc Fité, Jörg Jahnel, Davide Lombardo, Nicolas Mascot, Jeroen Sijsling, Andrew Sutherland, and John Voight.

Elliptic curves

$$E : y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Z}$$

Write $E_p := E \bmod p$, for p a prime of good reduction

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Theorem (Hasse)

$$\#E_p = p + 1 - a_p, \quad a_p \in [-2\sqrt{p}, 2\sqrt{p}]$$

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Theorem (Hasse)

$$\#E_p = p + 1 - a_p, \quad a_p \in [-2\sqrt{p}, 2\sqrt{p}]$$

Question

What can we say about the error term $a_p/\sqrt{p}$ as $p \to \infty$?

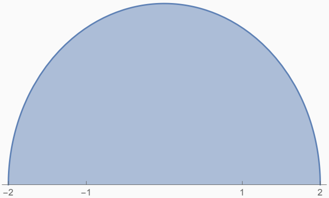
Two types of elliptic curves

$$a_p := p + 1 - \#E_p = \operatorname{tr} \mathrm{Frob}_p \in [-2\sqrt{p}, 2\sqrt{p}]$$

There are two limiting distributions for $a_p/\sqrt{p}$

non-CM

$\mathrm{End}_{\mathbb{Q}} E^{\mathrm{al}} = \mathbb{Q}$



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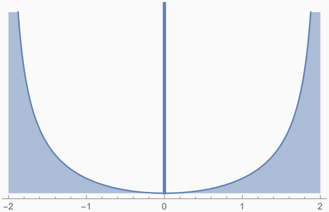

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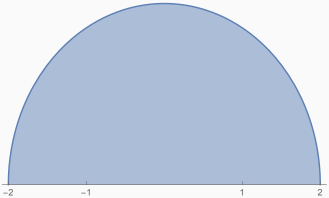

$\operatorname{Prob}(a_p = 0) \overset{?}{\sim} 1/\sqrt{p}$

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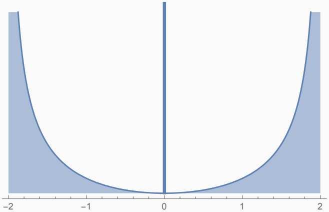

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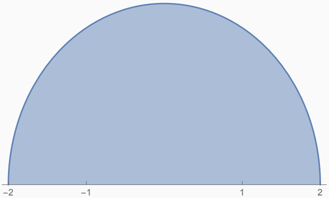

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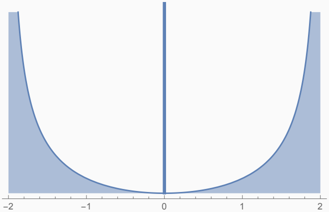

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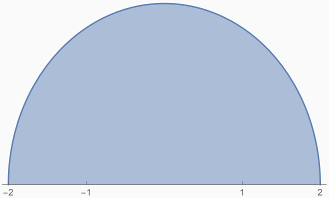

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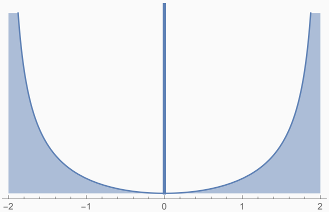

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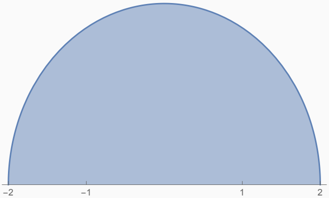

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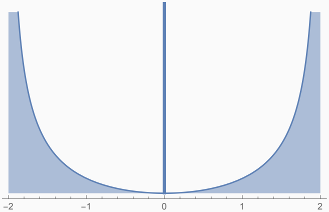

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How to distinguish between the two types?

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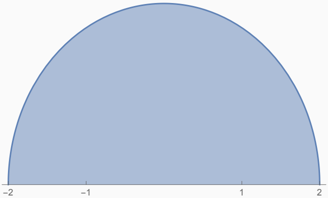

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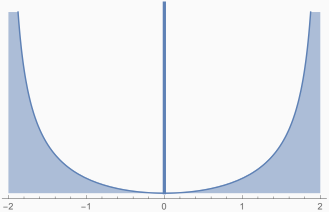

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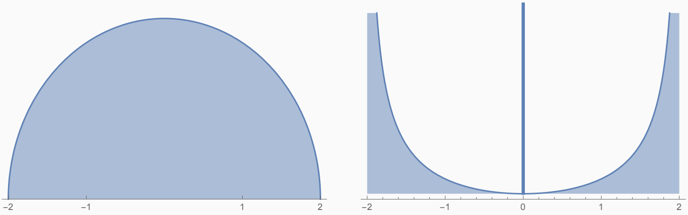

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- If E has CM, then

$$a_p \equiv 0 \bmod p \Leftrightarrow p \text{ inert or ramified in } \mathbb{Q}(\sqrt{-d})$$
$$\Leftrightarrow \mathrm{End}_{\mathbb{Q}} E^{\mathrm{al}} \not\simeq \mathrm{End}_{\mathbb{Q}} E_p^{\mathrm{al}}$$

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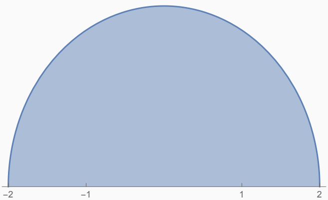

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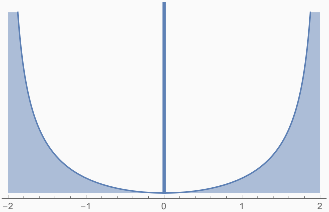

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- If E is non-CM, then $\mathrm{End}_{\mathbb{Q}} E_p^{\mathrm{al}} \cap \mathrm{End}_{\mathbb{Q}} E_q^{\mathrm{al}} \simeq \mathbb{Q}$ with prob. 1 and we expect $\mathrm{Prob}(a_p \equiv 0 \bmod p) \sim 1/\sqrt{p}$

Examples

$E : y^2 + y = x^3 - x^2 - 10x - 20$ **(11.a2)**

- $\mathrm{End}_{\mathbb{Q}} E_2^{\mathrm{al}} \simeq \mathbb{Q}(\sqrt{-1})$
- $\mathrm{End}_{\mathbb{Q}} E_3^{\mathrm{al}} \simeq \mathbb{Q}(\sqrt{-11})$
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$E: y^2 + y = x^3 - 7$ **(27.a2)**

- $p = 2 \bmod 3 \Rightarrow a_p = 0 \Rightarrow \operatorname{End}_{\mathbb{Q}} E_p^{\mathrm{al}}$ is a Quaternion algebra
- $p = 1 \bmod 3 \Rightarrow \operatorname{End}_{\mathbb{Q}} E_p^{\mathrm{al}} \simeq \mathbb{Q}(\sqrt{-3})$
- $\rightsquigarrow \operatorname{End}_{\mathbb{Q}} E^{\mathrm{al}} = \mathbb{Q}(\sqrt{-3})$

Genus 2 curves/Abelian surfaces

$$A := \mathrm{Jac}(C : y^2 = a_6x^6 + \cdots + a_0)$$

There are 6 possibilities for the real endomorphism algebra:

Abelian surface	$\mathrm{End}_{\mathbb{R}} A^{\mathrm{al}}$
square of CM elliptic curve	$\mathrm{M}_2(\mathbb{C})$
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Can we distinguish between these by looking at $A \bmod p$?

Zeta functions and Frobenius polynomials

- $C/\mathbb{Q}$ a nice curve of genus g
- $A := \mathrm{Jac}(C)$
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$$Z_p(T) := \exp\left(\sum_{r=1}^{\infty} \#C(\mathbb{F}_{p^r})T^r/r\right) \in \mathbb{Q}(t)$$

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where $\deg L_p(T) = 2g$ and

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- $g = 1 \rightsquigarrow L_p(T) = 1 - a_pT + pT^2$
- $g = 2 \rightsquigarrow L_p(T) = 1 - a_{p,1}T + a_{p,2}T^2 - a_{p,1}pT^3 + p^2T^4$

$L_p(T)$ gives us a lot of information about $A_p := A \bmod p$

Endomorphism algebras over finite fields

Theorem (Tate)

Let A be an abelian variety over $\mathbb{F}_q$, given

$$\det(1 - t\,\mathrm{Frob}\,|H^1(A)),$$

we may compute

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Example

If $L_5(T) = 1 - 2T^2 + 25T^4$, then:

- all endomorphisms are defined over $\mathbb{F}_{25}$, and
- $A_{\mathbb{F}_{25}}$ is isogenous to a square of an elliptic curve
- $\mathrm{End}_{\mathbb{Q}} A^{\mathrm{al}} \simeq M_2(\mathbb{Q}(\sqrt{-6}))$

Example continued

$$A = \mathrm{Jac}(y^2 = x^5 - x^4 + 4x^3 - 8x^2 + 5x - 1) \quad (262144.d.524288.1)$$

For $p = 5$, $L_5(T) = 1 - 2T^2 + 25T^4$, and:

- all endomorphisms of A_5 are defined over $\mathbb{F}_{25}$
- $\det(1 - T\,\mathrm{Frob}_5^2 \,|H^1(A)) = (1 - 2T + 25T^2)^2$
- A_5 over $\mathbb{F}_{25}$ is isogenous to a square of an elliptic curve
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For $p = 7$, $L_7(T) = 1 + 6T^2 + 49T^4$, and:

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$\Rightarrow \mathrm{End}_{\mathbb{R}} A^{\mathrm{al}} \neq M_2(\mathbb{C})$

Same L_p, different approach

We could have looked at the Néron–Severi lattice.

$$\mathrm{End}_{\mathbb{Q}}(A)^{\dagger} \simeq \mathrm{NS}(A) \otimes \mathbb{Q} \qquad \mathrm{NS}(A^{\mathrm{al}}) \hookrightarrow \mathrm{NS}(A_p^{\mathrm{al}})$$

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- $\operatorname{rk} \mathrm{NS}(A_p^{\mathrm{al}}) \in \{2, 4, 6\}$

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Example

- $\operatorname{rk} \mathrm{NS}(A_5^{\mathrm{al}}) = \operatorname{rk} \mathrm{NS}(A_7^{\mathrm{al}}) = 4$
- $\operatorname{disc} \mathrm{NS}(A_5^{\mathrm{al}}) = -6 \bmod \mathbb{Q}^{\times 2}$
- $\operatorname{disc} \mathrm{NS}(A_7^{\mathrm{al}}) = -10 \bmod \mathbb{Q}^{\times 2}$

$\Rightarrow \operatorname{rk} \mathrm{NS}(A^{\mathrm{al}}) \leq 3$

Fact

By a theorem of Charles, we know that at some point this method will attain a tight upper bound for $\operatorname{rk} \mathrm{NS}(A^{\mathrm{al}})$.

Real endomorphisms algebras and Picard numbers

Abelian surface	$\operatorname{End}_{\mathbb{R}} A^{\mathrm{al}}$	$\operatorname{rk} \operatorname{NS}(A^{\mathrm{al}})$
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• QM abelian surface • square of non-CM elliptic curve	$\mathrm{M}_2(\mathbb{R})$	3
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product of CM and non-CM elliptic curves	$\mathbb{C} \times \mathbb{R}$	2
• RM abelian surface • product of non-CM elliptic curves	$\mathbb{R} \times \mathbb{R}$	2
generic abelian surface	$\mathbb{R}$	1

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- $A_K \sim \prod_{i=1}^{t} A_i^{n_i}$,
- A_i unique and simple up to isogeny (over K),
- $B_i := \operatorname{End}_{\mathbb{Q}} A_i$ central simple algebra over $L_i := Z(B_i)$,
- $\dim_{L_i} B_i = e_i^2$,
- $\operatorname{End}_{\mathbb{Q}} A_K = \prod_{i=1}^{t} M_{n_i}(B_i)$

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Theorem (C–Mascot–Sijsling–Voight, C–Lombardo–Voight)

If Mumford–Tate conjecture holds for A, then we can compute

- t
- $\{(e_i n_i, n_i \dim A_i)\}_{i=1}^{t}$
- L_i

This is practical and its done by counting points (=computing L_p)

Real endomorphisms algebras, $\{e_i n_i, n_i \dim A_i\}_{i=1}^t$, and $\dim L_i$

Abelian surface	$\mathrm{End}_{\mathbb{R}} A^{\mathrm{al}}$	tuples	$\dim L_i$
square of CM elliptic crv	$\mathrm{M}_2(\mathbb{C})$	$\{(2,2)\}$	2
• QM abelian surface	$\mathrm{M}_2(\mathbb{R})$	$\{(2,2)\}$	1
• square of non-CM elliptic crv			
• CM abelian surface	$\mathbb{C} \times \mathbb{C}$	$\{(1,2)\}$	4
• product of CM elliptic crv		$\{(1,1),(1,1)\}$	2, 2
CM $\times$ non-CM elliptic crvs	$\mathbb{C} \times \mathbb{R}$	$\{(1,1),(1,1)\}$	2, 1
• RM abelian surface	$\mathbb{R} \times \mathbb{R}$	$\{(1,2)\}$	2
• prod. of non-CM elliptic crv		$\{(1,1),(1,1)\}$	1, 1
generic abelian surface	$\mathbb{R}$	$\{(1,1)\}$	1

Example continued: QM vs (non-CM)^2

$A = \mathrm{Jac}(y^2 = x^5 - x^4 + 4x^3 - 8x^2 + 5x - 1)$ (262144.d.524288.1)

- $\mathrm{End}_{\mathbb{Q}} A_3^{\mathrm{al}} \simeq M_2(\mathbb{Q}(\sqrt{-3}))$
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Question

Write $B := \mathrm{End}_Q A^{\mathrm{al}}$ and assume that B is a quaternion algr. Can we guess $\mathrm{disc}\, B$?

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Using Sato–Tate distributions

Abelian surface	$\mathrm{End}_{\mathbb{R}} A$	$\mathrm{rk}\,\mathrm{NS}(A)$	$\mathrm{M}_2[a_1]$	$\mathrm{M}_1[a_2]$
CM^2	$\mathrm{M}_2(\mathbb{C})$	4	8	4
• QM abelian surface • $(\text{non-CM})^2$	$\mathrm{M}_2(\mathbb{R})$	3	4	3
• CM abelian surface • CM $\times$ CM	$\mathbb{C} \times \mathbb{C}$	2	4	2
CM $\times$ non-CM	$\mathbb{C} \times \mathbb{R}$	2	3	2
• RM abelian surf. • non-CM $\times$ non-CM	$\mathbb{R} \times \mathbb{R}$	2	2	2
generic abelian surface	$\mathbb{R}$	1	1	1

Question

Can you spot the pattern?

Arithmetic invariants from Sato-Tate moments

Theorem (Fité–C–Sutherland)

Let A be an abelian variety, then we ahve

- $\operatorname{rk}\operatorname{End}(A) = \mathrm{M}_2[a_1] = \mathrm{E}_{\mathrm{ST}_A}[\mathrm{tr}^2]$
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What else can we deduce about abelian varieties?

K3 surfaces

K3 surfaces are a possible generalization of elliptic curves

They may arise in many ways:

- smooth quartic surfaces in $\mathbb{P}^3$

$$X : f(x, y, z, w) = 0, \quad \deg f = 4$$

- double cover of $\mathbb{P}^2$ branched over a sextic curve

$$X : w^2 = f(x, y, z), \quad \deg f = 6$$

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Can we play similar game as before?

In this case, instead of studying $\# X_p$ or $\operatorname{tr} \mathrm{Frob}_p$ we study

$$p \longmapsto \operatorname{rk} \mathrm{NS}\, {X_p}^{\mathrm{al}} \in \{2, 4, \ldots, 22\}$$

K3 Surfaces

$X/\mathbb{Q}$ a K3 surface

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Recall that:

- $\operatorname{rk} \operatorname{End} {E_p}^{\mathrm{al}} = 4 \iff a_p = 0$
- $\operatorname{Prob}(a_p = 0) = \begin{cases} \overset{?}{\sim} \frac{1}{\sqrt{p}} & \text{if } E \text{ is non-CM (Lang–Trotter)} \\ 1/2 & \text{if } E \text{ has CM by } \mathbb{Q}(\sqrt{-d}) \end{cases}$

In the later case,

$$\{p : a_p = 0\} = \{p : p \text{ is ramified or inert in } \mathbb{Q}(\sqrt{-d})\}$$

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For an abelian surface A we have:

$$\mathrm{NS}(A)_{\mathbb{Q}} \simeq \{\phi \in \mathrm{End}(A)_{\mathbb{Q}} : \phi^{\dagger} = \phi\},$$

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where † denotes the Rosati involution. Thus for $A/\mathbb{Q}$ this is equivalent to

$$p \longmapsto \operatorname{rk} \operatorname{End}(A_p{}^{\mathrm{al}})^{\dagger}_{\mathbb{Q}} = \operatorname{rk} \operatorname{NS} A_p{}^{\mathrm{al}} \in \{2, 4, 6\}$$

Now

- $\operatorname{rk} \operatorname{NS} A_p{}^{\mathrm{al}} \geq 4 \iff A_p{}^{\mathrm{al}} \sim E^2$
- $\operatorname{rk} \operatorname{NS} A_p{}^{\mathrm{al}} = 6 \iff A_p{}^{\mathrm{al}} \sim E^2$, E supersingular, i.e., $a_p = 0$

Néron–Severi group

- $\mathrm{NS}\bullet$ = Néron–Severi group of $\bullet \simeq \{\text{curves on } \bullet\}/\sim$
- $\rho(\bullet) = \operatorname{rk} \mathrm{NS}\bullet$
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Theorem (Charles)

For infinitely many p we have $\rho(X_p{}^{\mathrm{al}}) = \min_q \rho(X_q{}^{\mathrm{al}})$.

The Problem

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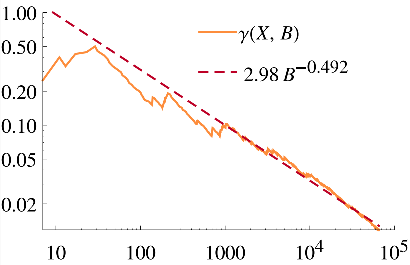
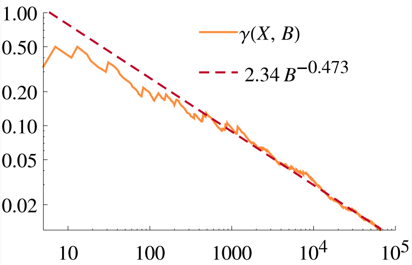
Let's do some numerical experiments!

Two generic K3 surfaces with $\rho(X^{\mathrm{al}}) = 1$





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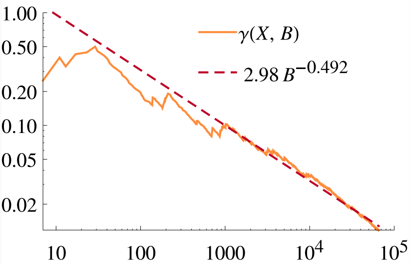

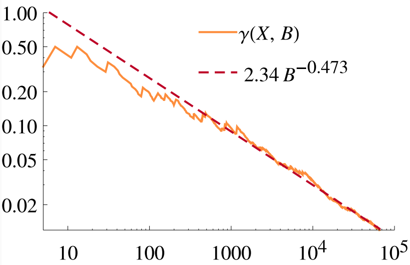

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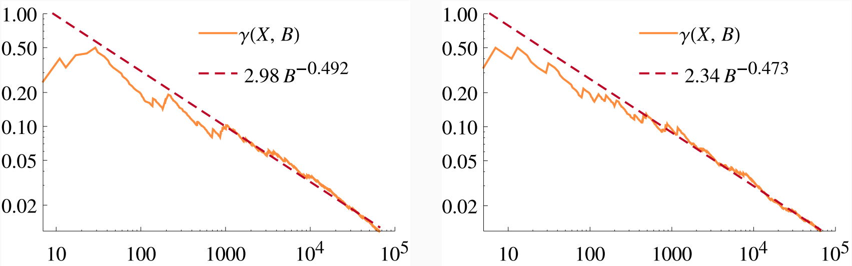

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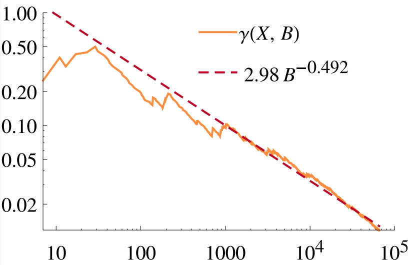

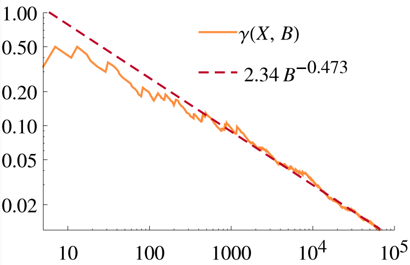

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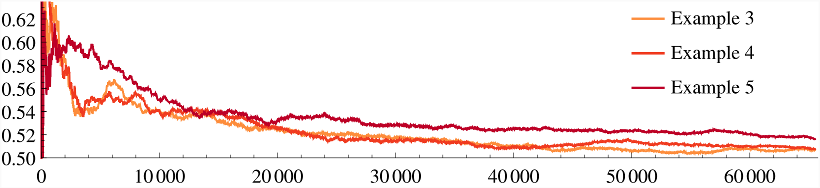
Why?

Three K3 surfaces with $\rho(X^{al}) = 2$



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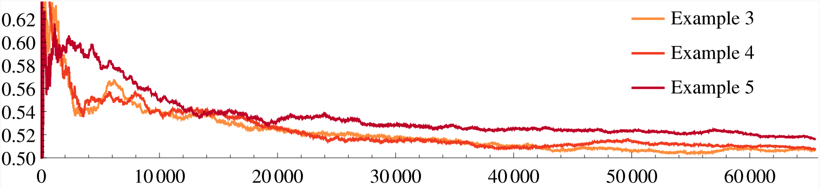

No obvious trend...

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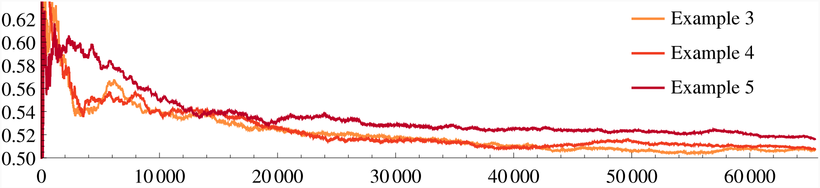

No obvious trend...

Could it be related to some integer being a square modulo p?

We can explain the $1/2$

Theorem (C, C–Elsenhans–Jahnel)

If $\rho(X^{\mathrm{al}}) = \min_q \rho(X_p{}^{\mathrm{al}})$, then there is a $d_X \in \mathbb{Z}$ such that:

$$\left\{p > 2 : p \text{ inert in } \mathbb{Q}(\sqrt{d_X})\right\} \subset \Pi_{\mathrm{jump}}(X).$$

In general, d_X is not a square.

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Corollary

If d_X is not a square:

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$D_3 = -1 \cdot 5 \cdot 151 \cdot 22490817357414371041 \cdot 3873084974301493372336663588079962607808750567408509842132769703432789353$

$D_4 = 53 \cdot 2624174618795407 \cdot 512854561846964817139494202072778341 \cdot 121521837008902876907671810212692174435336 2873 \cdot 68$

$D_5 = -1 \cdot 47 \cdot 3109 \cdot 4969 \cdot 14857095849982608071 \cdot 445410277660928347762586764331874432202584688016149 \cdot 65865270852$

Experimental data for $\rho(X^{\mathrm{al}}) = 2$ (again)

What if we ignore $\{p > 2 : p \text{ inert in } \mathbb{Q}(\sqrt{d_X})\} \subset \Pi_{\mathrm{jump}}(X)$?

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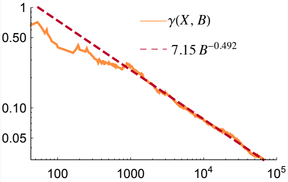
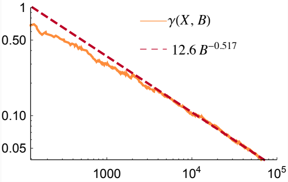
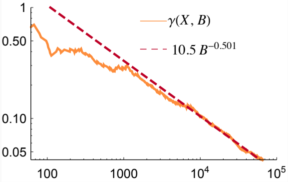
$$\gamma\left(X_{\mathbb{Q}(\sqrt{d_X})}, B\right) \overset{?}{\sim} \frac{c}{\sqrt{B}}, \quad B \to \infty$$



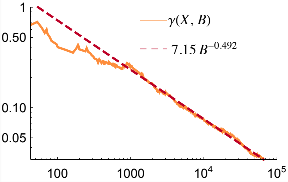


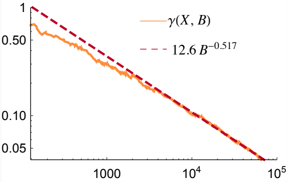


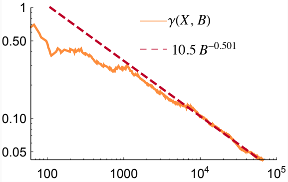
Experimental data for $\rho(X^{\mathrm{al}}) = 2$ (again)

What if we ignore $\{p > 2 : p \text{ inert in } \mathbb{Q}(\sqrt{d_X})\} \subset \Pi_{\mathrm{jump}}(X)$?

$$\gamma\left(X_{\mathbb{Q}(\sqrt{d_X})}, B\right) \overset{?}{\sim} \frac{C}{\sqrt{B}}, \quad B \to \infty$$







$$\mathrm{Prob}(p \in \Pi_{\mathrm{jump}}(X)) = \begin{cases} 1 & \text{if } d_X \text{ is not a square modulo } p \\ \overset{?}{\sim} \frac{1}{\sqrt{p}} & \textit{otherwise} \end{cases}$$

Why?!?