Computing isogeny classes of genus 2 Jacobians over $\mathbb{Q}$

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Slides available at edgarcosta.org

Joint work with Raymond van Bommel, Shiva Chidambaram, and Jean Kieffer.

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Idea: Two abelian varieties are *isogenous* if one is a "finite quotient" of the other.

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What shapes can these graphs take?

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  (Chiloyan–Lozano-Robledo 2021) That is all, LMFDB has all the possibilities.

Elliptic curves: complex analytic picture

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An ℓ-isogeny $E_\tau \to E_{\tau'}$ corresponds to choosing a cyclic subgroup $K \subset E_\tau[\ell]$ of order ℓ.

There are $\ell + 1$ such subgroups, giving $\ell + 1$ choices for τ':

$$\tau \mapsto \ell\tau \quad \text{or} \quad \tau \mapsto \frac{\tau + j}{\ell} \text{ for } j = 0, 1, \ldots, \ell - 1$$

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Key point: Isogenies $\longleftrightarrow$ finite subgroups of torsion.

This generalizes to abelian varieties of any dimension.

Abelian surfaces

Very little is known beyond elliptic curves over $\mathbb{Q}$.

www.LMFDB.org has genus 2 curves with small minimal absolute discriminant.

These are grouped by isogeny class of their Jacobian.

However, the isogeny classes are known to not be complete.

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Given an abelian surface A, compute its isogeny class.

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1. List irreducible isogeny types.
2. List the possible degrees for each type.
3. Search for all isogenies of a given type and degree.
4. Reapply as needed.

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Irreducible isogenies have kernel $K \subset A[\ell]$ or $K \subset A[\ell^2]$, maximal isotropic.

- 1-step: $K \subset A[\ell]$, degree ℓ^2
- 2-step: $K \subset A[\ell^2]$ with $K \simeq (\mathbb{Z}/\ell\mathbb{Z})^2 \times \mathbb{Z}/\ell^2\mathbb{Z}$, degree ℓ^4

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Key difference: For surfaces, we must also consider kernels in $A[\ell^2]$, not just $A[\ell]$.

Irreducible isogeny types for typical surfaces

If $\mathrm{End}(A)^\dagger = \mathbb{Z}$, then an isogeny $\varphi : A \to B$ can always be factored as

$$A \xrightarrow{\alpha \in \mathrm{End}(A)} A \xrightarrow{\varphi_1} A_1 \xrightarrow{\varphi_2} A_2 \xrightarrow{\varphi_3} \cdots \xrightarrow{\varphi_n} A_n = B,$$

where $\deg \varphi_i \in \{\ell_i^{\dim A}, \ell_i^{2\dim A}\}$ for ℓ_i prime.

Furthermore, if $\dim A = 2$, we can assure that $K_i = \ker \varphi_i$ is:

- 1-step: maximal isotropic subgroup of $A[\ell_i]$, or
- 2-step: maximal isotropic subgroup of $A[\ell_i^2]$ and $K_i \simeq (\mathbb{Z}/\ell_i\mathbb{Z})^2 \times \mathbb{Z}/\ell_i^2\mathbb{Z}$.

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Given a genus 2 curve C, compute all other curves whose Jacobians are isogenous to $\mathrm{Jac}(C)$ over $\mathbb{Q}$.

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   Typical surface: maximal isotropic subgroups of $A[\ell^2]$ are also a possibility, i.e., kernels of size ℓ^2 or ℓ^4.
2. List the possible degrees for each type.
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Possible degrees

Theorem (Mazur)

Let ℓ be a prime such that there exists an isogeny $\varphi : E \to E'$ with kernel of order ℓ.

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For $\mathrm{End}(A^{\mathrm{al}}) = \mathbb{Z}$ we can instead do one surface at a time (Dieulefait).

Algorithm (Dieulefait)[1]

Input: Conductor of A and a finite list of L-polynomials

Output: Finite superset of primes ℓ with reducible mod-ℓ Galois representation.

In particular, this gives the primes ℓ for which 1-step or 2-step ℓ-isogenies are possible.

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Example

$$C\colon y^2 + (x+1)y = x^5 + 23x^4 - 48x^3 + 85x^2 - 69x + 45$$

the only possibilities are isogenies of degree 31^2.

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Searching for isogenies

For elliptic curves, one may use modular polynomials $\phi_\ell(x, y) \in \mathbb{Z}[x, y]$. Defined by

$$\phi_\ell(j, j') = 0 \iff \exists\, \varphi : E_j \longrightarrow E_{j'} \text{ such that } \ker \varphi \simeq \mathbb{Z}/\ell\mathbb{Z}$$

The size grows as $\widetilde{O}(\ell^3)$

- $\ell = 17$: 23 KB, 8 pages 📜
- $\ell = 163$: 28 MB, 5000+ pages 📚

Modular polynomials for surfaces are impractical!

More variables $\phi_\ell(x_1, x_2, x_3, y) \in \mathbb{Z}[x_1, x_2, x_3, y]$.

Size grows as $\widetilde{O}(\ell^{15})$.

- $\ell = 2$: 1.4 MB 💾
- $\ell = 3$: 400 MB 💿

We will instead use complex analytic methods.

Geometric invariants

Elliptic curves

$E : y^2 = x^3 - 27c_4x - 54c_6$

- $c_4, c_6 \in \mathbb{Z}$ — **integral** invariants
- $j(E) = 1728\frac{c_4^3}{c_4^3 - c_6^2}$ — a **ratio**, not always integral

Over $\mathbb{C}$: $E \simeq \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, and c_4, c_6 come from modular forms $E_4(\tau), E_6(\tau)$.

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$C : y^2 = f(x)$, with Igusa–Clebsch invariants (I_2, I_4, I_6, I_{10})

- $I_4, I_{10} \in \mathbb{Z}$ — **integral** (like c_4, c_6)
- I_2, I_6 — only **ratios** of modular forms (like j!)

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Problem: We need invariants that are true modular forms (not ratios).

Choice of generators

Solution: Use Siegel modular forms M_4, M_6, M_{10}, M_{12} instead.

These are related to the Igusa–Clebsch invariants, e.g., $M_4 = 2^{-2} I_4$, $M_{10} = -2^{-12} I_{10}$.

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Key properties

- $(M_4, M_6, M_{10}, M_{12})$ generate all Siegel modular forms
- They have **integer** Fourier coefficients
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Theorem (Igusa)

If f is a Siegel modular form of even weight k with integer Fourier coefficients, then $12^k f \in \mathbb{Z}[M_4, M_6, M_{10}, M_{12}]$.

Analogy: M_k play the role of c_4, c_6 (integral), not j (ratio).

Combining complex analytic and algebraic methods

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Theorem (Kieffer 2022)

Assume that there exists $\lambda \in \mathbb{C}^\times$ such that $\lambda^k M_k(\tau) \in \mathbb{Z}$.
If f is a Siegel modular form of even weight k with integer Fourier coefficients,

$$\prod_{\gamma} \left(X - (12\lambda\ell^{c_\gamma})^k f(\gamma\tau) \right)$$

has **integer** coefficients, where γ loops over specific coset representatives for the Hecke operator $T(\ell)$ (resp. $T_1(\ell^2)$) and $0 \le c_\gamma \le 2$ (resp. 3).

$$\left\{\mathbb{C}^2/\left(\mathbb{Z}^2 + \gamma\tau\mathbb{Z}^2\right)\right\}_\gamma = \left\{\text{surfaces 1-step (resp. 2-step) isogenous to } \mathbb{C}^2/(\mathbb{Z}^2 + \tau\mathbb{Z}^2)\right\}$$

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In other words, if we start with $\lambda^k M_k(\tau) \in \mathbb{Z}$, then

$$(12\lambda \ell^{c_\gamma})^k M_k(\gamma\tau_C)$$

can be grouped in Galois orbits of algebraic **integers**.

Complex analytic approach

Given $(m_4, m_6, m_{10}, m_{12}) \in \mathbb{P}(4,6,10,12)(\mathbb{Z})$ and ℓ.
Compute complex balls that provably contain:

1. $\tau \in \mathbb{H}_2$
2. $\lambda \in \mathbb{C}^\times$ such that $\lambda^k M_k(\tau) = m_k$
3. For each coset representative γ of the Hecke operator $T(\ell)$ (or $T_1(\ell^2)$)

$$(12\lambda\ell^{c_\gamma})^k M_k(\gamma\tau).$$

Keep the γ's such that the computed balls for $\left(12\lambda\ell^{c_\gamma}\right)^k M_k(\gamma\tau)$ contain an integer.

Complex approach

For $\ell = 31$ and $C\colon y^2 + (x+1)y = x^5 + 23x^4 - 48x^3 + 85x^2 - 69x + 45$ there is only one γ such that the ball $\left(12\lambda\ell^{c_\gamma}\right)^4 M_4(\gamma\tau) \cap \mathbb{Z} \neq \emptyset$, and

$$(12\lambda\ell^{c_\gamma})^4 M_4(\gamma\tau) = \alpha^2 \cdot 318972640 \pm 7.8 \times 10^{-47}$$
$$(12\lambda\ell^{c_\gamma})^6 M_6(\gamma\tau) = \alpha^3 \cdot 1225361851336 \pm 5.5 \times 10^{-39}$$
$$(12\lambda\ell^{c_\gamma})^{10} M_{10}(\gamma\tau) = \alpha^5 \cdot 10241530643525839 \pm 1.6 \times 10^{-29}$$
$$(12\lambda\ell^{c_\gamma})^{12} M_{12}(\gamma\tau) = -\alpha^6 \cdot 307105165233242232724 \pm 4.6 \times 10^{-22}$$

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We can confirm that these are indeed integers by certifying the vanishing of

$$\prod_{\gamma} \left(\left(12\lambda\ell^{c_\gamma}\right)^k M_k(\gamma\tau) - m'_k \right) \in \mathbb{Z}.$$

by recomputing the relevant $\left(12\lambda\ell^{c_\gamma}\right)^k M_k(\gamma\tau)$ at higher precision.

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Open: Certificates for completeness of isogeny graphs.

LMFDB

Originally 63 107 typical genus 2 curves, split amongst 62 600 isogeny classes.

By computing isogeny classes, we found 21 923 new curves.

Only 2 523 new curves are explained by Richelot isogenies.

Size	1	2	3	4	5	6	7	8	9	10	12	16	18
Count	51 549	2 672	6 936	420	756	164	40	45	3	2	3	9	1

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The whole computation took 75 hours. Only 3 classes took more than 10 minutes:

- 349.a 56 min, found isogeny of degree 13^4.
- 353.a 23 min, found isogeny of degree 11^4.
- 976.a 19 min, checking that no isogeny of degree 29^4 exists.

Coming soon to LMFDB

There is a new set of 5 235 806 curves soon to be added to LMFDB.

Of these, 1 823 592 are typical, split amongst 1 538 149 isogeny classes.

We found $687\,763+\varepsilon$ new curves (in 97 days).

Of those 289 553 could be obtained via Richelot isogenies.

Size	1	2	3	4	5	6	7	8	≥ 9
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We discovered irreducible isogenies of degree

$$\{2^2, 3^2, 2^4, 5^2, 7^2, 3^4, 13^2, 17^2, 5^4, 31^2, 7^4, 11^4, 13^4\}.$$

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Some observations per size:

- 2: 75% degree 2^2, 22% degree 3^4, and then 3^2, 5^4, 5^2, 7^4, 7^2, ...
- 3: 99.2% are $\triangle$ made up of degree 2^4 isogenies.
- 4: 97.8% are $\succ\!\!-$ made up of degree 2^2 isogenies.
- 5: 99.8% are $\bowtie$ made up of degree 2^4 isogenies.
- 6: 75% + 15% are graphs made up of degree 2^2 isogenies.

Life, the universe, and everything

42 Richelot isogenous curves with conductor $497051100 = 2^2 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17^2$