Variation of Néron–Severi ranks of reductions of K3 surfaces

Edgar Costa (MIT)

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Joint Mathematics Meetings

Slides available at `edgarcosta.org` under Research

Elliptic curves

$$E : y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Z}$$

Write $E_p := E \bmod p$

- What can we say about $\#E_p$ for an arbitrary p?
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$\rightsquigarrow$ studying the **statistical** properties $\#E_p$.

Hasse's bound

Theorem (Hasse)

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Question

What can we say about the error term $a_p/\sqrt{p}$ as $p \to \infty$?

Two types of elliptic curves

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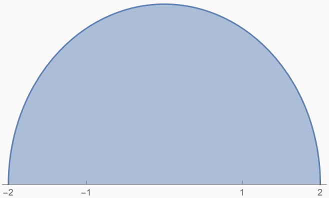

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non-CM	CM
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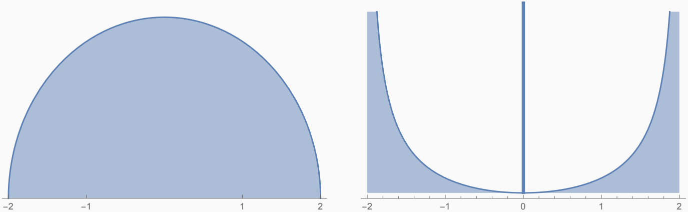

Distinguishing between the two types

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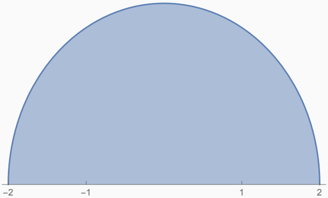

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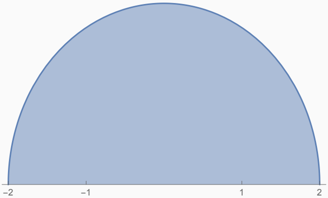

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$$a_p \equiv 0 \bmod p \Leftrightarrow p \text{ inert or ramified in } \mathbb{Q}(\sqrt{-d})$$

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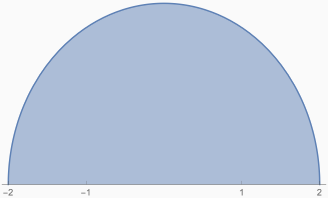

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$$\Leftrightarrow \mathrm{End}_{\mathbb{Q}} E^{\mathrm{al}} \not\simeq \mathrm{End}_{\mathbb{Q}} E_p^{\mathrm{al}}$$
- If E is non-CM, then $\mathrm{End}_{\mathbb{Q}} E_p^{\mathrm{al}} \cap \mathrm{End}_{\mathbb{Q}} E_q^{\mathrm{al}} \simeq \mathbb{Q}$ with prob. 1 and we expect $\mathrm{Prob}(a_p \equiv 0 \bmod p) \sim 1/\sqrt{p}$

K3 surfaces

K3 surfaces are a possible generalization of elliptic curves

They may arise in many ways:

- smooth quartic surfaces in $\mathbb{P}^3$

$$X : f(x, y, z, w) = 0, \quad \deg f = 4$$

- double cover of $\mathbb{P}^2$ branched over a sextic curve

$$X : w^2 = f(x, y, z), \quad \deg f = 6$$

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Can we play similar game as before?

In this case, instead of studying $\# X_p$ or $\operatorname{Tr} \operatorname{Frob}_p$ we study

$$p \longmapsto \operatorname{rk} \operatorname{NS} {X_p}^{\mathrm{al}} \in \{2, 4, \ldots, 22\}$$

K3 Surfaces

$X/\mathbb{Q}$ a K3 surface

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This is analogous to studying:

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Recall that:

- $\operatorname{rk} \operatorname{End} {E_p}^{\mathrm{al}} = 4 \iff a_p \equiv 0 \bmod p$
- $\operatorname{Prob}(a_p = 0) = \begin{cases} \overset{?}{\sim} \frac{1}{\sqrt{p}} & \text{if } E \text{ is non-CM (Lang–Trotter)} \\ 1/2 & \text{if } E \text{ has CM by } \mathbb{Q}(\sqrt{-d}) \end{cases}$

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In the later case,

$$\{p : a_p \equiv 0 \bmod p\} = \{p : p \text{ is ramified or inert in } \mathbb{Q}(\sqrt{-d})\}$$

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For an abelian surface A we have:

$$\operatorname{NS}(A)_{\mathbb{Q}} \simeq \{\phi \in \operatorname{End}(A)_{\mathbb{Q}} : \phi^{\dagger} = \phi\},$$

where † denotes the Rosati involution.

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where † denotes the Rosati involution. Thus for $A/\mathbb{Q}$ this is equivalent to

$$p \longmapsto \operatorname{rk} \operatorname{End}({A_p}^{\mathrm{al}})^{\dagger}_{\mathbb{Q}} = \operatorname{rk} \operatorname{NS} {A_p}^{\mathrm{al}} \in \{2, 4, 6\}$$

Now

- $\operatorname{rk} \operatorname{NS} {A_p}^{\mathrm{al}} \geq 4 \iff {A_p}^{\mathrm{al}} \sim E^2$
- $\operatorname{rk} \operatorname{NS} {A_p}^{\mathrm{al}} = 6 \iff {A_p}^{\mathrm{al}} \sim E^2$, E supersingular, i.e., $a_p = 0$

Néron–Severi group

- $\operatorname{NS} \bullet$ = Néron–Severi group of $\bullet \simeq \{\text{curves on } \bullet\}/\sim$
- $\rho(\bullet) = \operatorname{rk} \operatorname{NS} \bullet$
- $X_p := X \bmod p$

$$\begin{array}{ccccccc}
X & \longrightarrow & \operatorname{NS} X^{\mathrm{al}} & \longrightarrow & \rho(X^{\mathrm{al}}) & & \in \{1, 2, \ldots, 20\} \\
\big\downarrow & & \big\downarrow & & \big\updownarrow \text{???} & & \\
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For infinitely many p we have $\rho(X_p{}^{\mathrm{al}}) = \min_q \rho(X_q{}^{\mathrm{al}})$.

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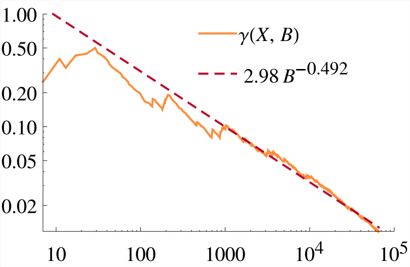
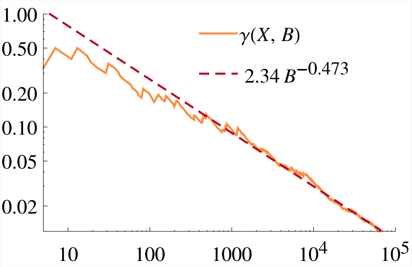
Let's do some numerical experiments!

Two generic K3 surfaces, $\rho(X^{\mathrm{al}}) = 1$





$$\gamma(X, B) \overset{?}{\sim} \frac{c_X}{\sqrt{B}}, \quad B \to \infty$$

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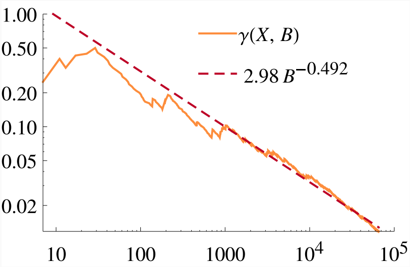

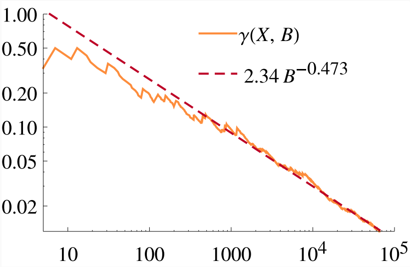

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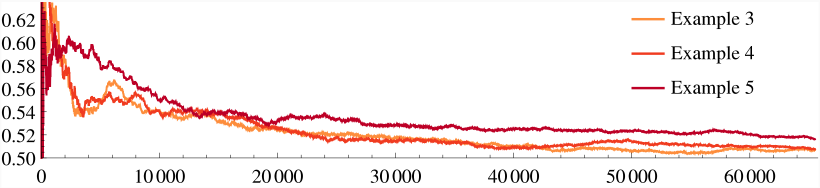
Why?

Three K3 surfaces with $\rho(X^{al}) = 2$



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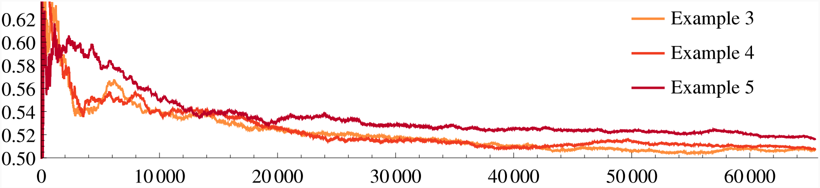

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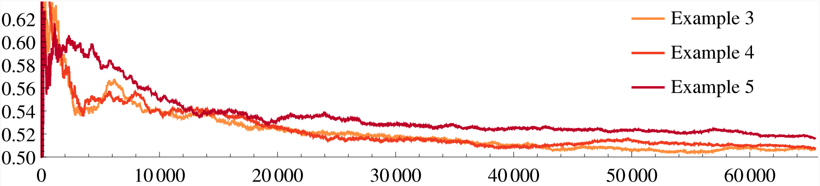

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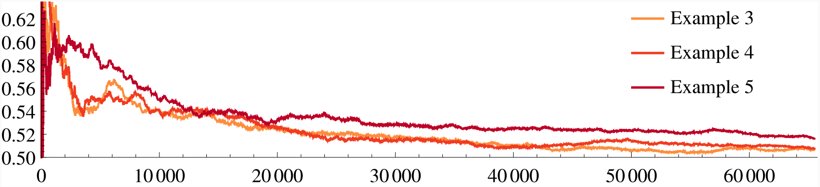
Could it be related to some integer being a square modulo p?

We can explain the 1/2



Theorem (C, C–Elsenhans–Jahnel)

If $\rho(X^{\mathrm{al}}) = \min_q \rho({X_p}^{\mathrm{al}})$, *then there is a* $d_X \in \mathbb{Z}$ *such that:*

$$\left\{ p > 2 : p \textit{ inert in } \mathbb{Q}(\sqrt{d_X}) \right\} \subset \Pi_{\mathrm{jump}}(X).$$

In general, d_X *is not a square.*

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$D_3 = -1 \cdot 5 \cdot 151 \cdot 22490817357414371041 \cdot 3873084974301493372336663588079962607808750567408509842132769703432789353$

$D_4 = 53 \cdot 2624174618795407 \cdot 512854561846964817139494202072778341 \cdot 1215218370089028769076718102126921744353362873 \cdot 68$

$D_5 = -1 \cdot 47 \cdot 3109 \cdot 4969 \cdot 14857095849982608071 \cdot 44541027766092834776258676433187443220258468801 6149 \cdot 65865270852$

Experimental data for $\rho(X^{\mathrm{al}}) = 2$ (again)

What if we ignore $\{p > 2 : p \text{ inert in } \mathbb{Q}(\sqrt{d_X})\} \subset \Pi_{\mathrm{jump}}(X)$?

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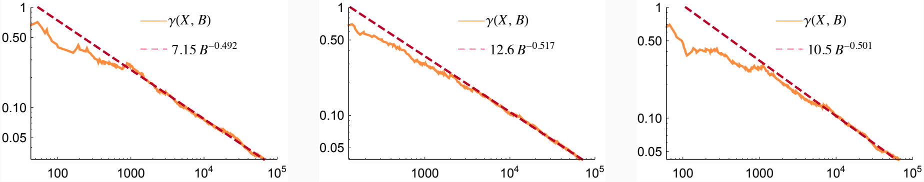
$$\gamma\left(X_{\mathbb{Q}(\sqrt{d_X})}, B\right) \overset{?}{\sim} \frac{c_X}{\sqrt{B}}, \quad B \to \infty$$

Example 3 | Example 4 | Example 5



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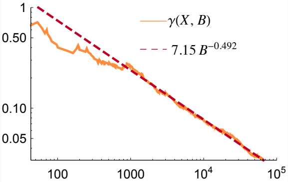

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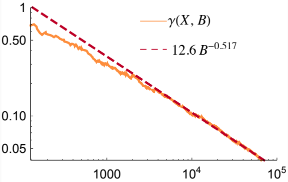
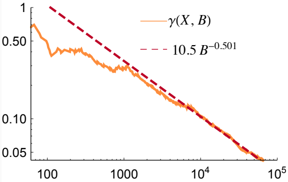


Example 5



$$\mathrm{Prob}(p \in \Pi_{\mathrm{jump}}(X)) = \begin{cases} 1 & \text{if } d_X \text{ is not a square modulo } p \\ \overset{?}{\sim} \frac{1}{\sqrt{p}} & \textit{otherwise} \end{cases}$$

Why?