Computing isogeny classes of typical principally polarized abelian surfaces over the rationals

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Joint work with Raymond van Bommel, Shiva Chidambaram, and Jean Kieffer.

A more informal title

Computing isogeny classes of typical principally polarized abelian surfaces over the rationals

- isogeny class = "friendship graph"
- typical = "ordinary"
- principally polarized = "popular"
- typical + principally polarized + surface $\Rightarrow \text{Jac}(\text{genus 2 curve})$

Computing friendship graphs of ordinary genus 2 curves over the rationals

Isogeny classes

Definition

An isogeny between two abelian varieties is a $\varphi : A \twoheadrightarrow B$ such that $\# \ker \varphi < \infty$.

The isogeny class is obtained by taking quotients by finite rational subgps.

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- L-function
- Rank
- Endomorphism algebra

In particular, the class must be finite (Faltings 1983).

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What shape can these take?

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  (Chiloyan–Lozano-Robledo 2021) That is all, LMFDB has all the possibilities.

Abelian surfaces

Very little is know away from elliptic curves over $\mathbb{Q}$.

www.LMFDB.org has genus 2 curves grouped by isogeny class of their Jacobian.

However, the isogeny classes are not complete.

Problem

Given an abelian surface A compute its isogeny class.

Generic approach

1. List irreducible isogeny types
2. Bound the prime divisors of their degree
3. Search for all isogenies of a given type and degree.

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   We will instead use analytical methods.

Analytic approach

Input: Genus 2 curve C

Output: All genus 2 curves such that their Jacobians are isogenous to C

- Dieulefait's tests tell us that degrees to consider. For

$$C\colon y^2 + (x+1)y = x^5 + 23x^4 - 48x^3 + 85x^2 - 69x + 45.$$

  the only possibility is 31^2.
- Compute $\tau \in \mathbb{H}_2$ as a ball that represents the isomorphism class of C.

Theorem

There exist M_k with $k \in 4, 6, 10, 12$ of weight k with integral Fourier coefficients that generate the graded $\mathbb{C}$-algebra of Siegel modular forms of $\mathbb{H}_2$.

- Compute $\lambda \in C^\times$ such that $\lambda^k M_k(\tau) = M_k(C)$ (via Igusa–Clebsch invariants) λ accounts for converting a big period matrix to a small period matrix $\tau \in \mathbb{H}_2$.

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- Compute $\lambda \in C^\times$ such that $\lambda^k M_k(\tau) = M_k(C)$ (via Igusa–Clebsch invariants)
- We now loop over all coset representative γ of the Hecke operator $T(\ell)$ (or $T_1(\ell^2)$) and compute $\left(\lambda c_\gamma\right)^k M_k(\gamma\tau) \in \mathbb{C}$ as balls.
  The constant c_γ such that they form Galois orbits of algebraic integers.
- Keep the γ's such that the computed balls for $\left(\lambda c_\gamma\right)^k M_k(\gamma\tau)$ contain an integer.

Analytic approach

- Keep the γ's such that the computed balls for $(\lambda c_\gamma)^k M_k(\gamma\tau)$ contain an integer.
  For our example, there is only one such γ, and
  $$(\lambda c_\gamma)^4 M_4(\gamma\tau) = \alpha^2 \cdot 318972640 \pm 7.8 \times 10^{-47}$$
  $$(\lambda c_\gamma)^6 M_6(\gamma\tau) = \alpha^3 \cdot 1225361851336 \pm 5.5 \times 10^{-39}$$
  $$(\lambda c_\gamma)^{10} M_{10}(\gamma\tau) = \alpha^5 \cdot 10241530643525839 \pm 1.6 \times 10^{-29}$$
  $$(\lambda c_\gamma)^{12} M_{12}(\gamma\tau) = -\alpha^6 \cdot 307105165233242232724 \pm 4.6 \times 10^{-22}$$
  where $\alpha = 2^2 \cdot 3^2 \cdot 31$.
- Recompute $(\lambda c_\gamma)^k M_k(\gamma\tau)$ with enough precision to certify the vanishing of
  $$\prod_\gamma ((\lambda c_\gamma)^k M_k(\gamma\tau) - m'_k) \in \mathbb{Z}.$$
- Reapply the method to the new invariants obtained.

Reconstructing curves

In our example, the isogeny class only contains one other curve.

We find it by first applying Mestre's algorithm to obtain

$$C' : y^2 = -1624248x^6 + 5412412x^5 - 6032781x^4 + 876836x^3 - 1229044x^2 - 5289572x - 1087304,$$

a quadratic twist by -83761 of the desired curve

$$C'' : y^2 + xy = -x^5 + 2573x^4 + 92187x^3 + 2161654285x^2 + 406259311249x + 93951289752862.$$

Computing the isogeny class of this example took 113 minutes of CPU time.

Almost all of the time is spent on certifying the results.

One can independently obtain a certificate for the isogeny (6.5 hours and 3 MB).

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We would like to do the same for the completeness of the isogeny graph.

LMFDB

We ran our algorithm on LMFDB. The whole computation took 75 hours of CPU time.

Originally 63 107 typical genus 2 curves, split amongst 62 600 isogeny classes.

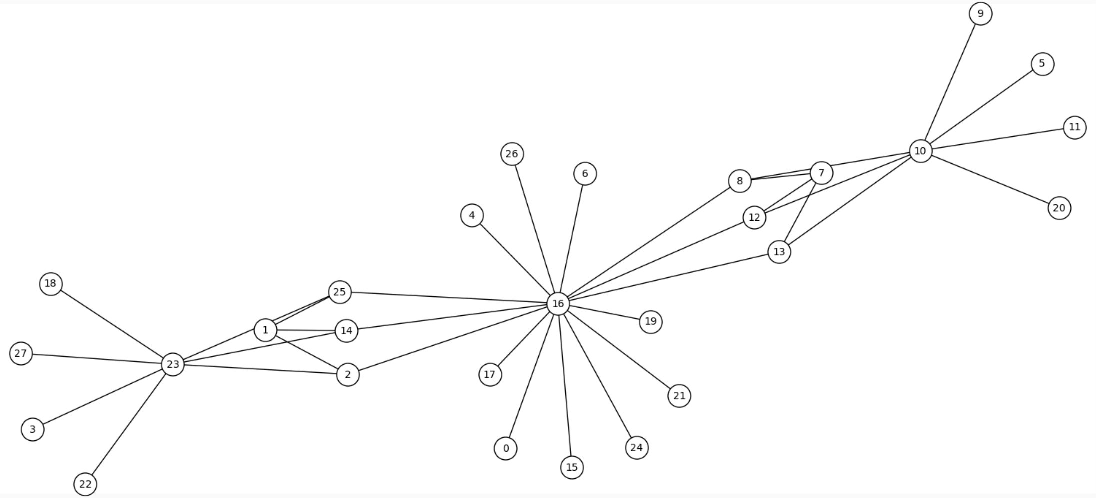
By computing isogeny classes, we found 21 923 new curves.

Size	1	2	3	4	5	6	7	8	9	10	12	16	18
Count	51549	2672	6936	420	756	164	40	45	3	2	3	9	1

Only 3 classes took more than 10 minutes.

- 349.a 56 min, found isogeny of degree 13^4.
- 353.a 23 min, found isogeny of degree 11^4.
- 976.a 19 min, checking that no isogeny of degree 29^4 exists.

Rank 28 graph



All Richolet isogenies.

3 isogeny classes with this graph with conductor $50274 = 2 \cdot 3^3 \cdot 7^2 \cdot 19$