Effective Computation of Hodge Cycles

Edgar Costa (MIT)

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Slides available at edgarcosta.org.

Joint work with Nicholas Mascot, Jeroen Sijsling, John Voight, and Emre Can Sertöz

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Let A be an abelian variety defined over k.

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  determines the k-isogeny class and the isomorphism class of $\operatorname{End}(A) \otimes \mathbb{Q}$.
  If $A = \operatorname{Jac}(C)$, then we can compute this by counting points on C.
- There are several in principle algorithms to do this over a number field.
  These involve, a day/night algorithm:
  - by day: search for reasonable morphisms;
  - by night: restrict your search space.

Our setup

Let C be a nice (smooth, projective, geometrically integral) curve over k of genus g given by equations. Let J be the Jacobian of C.

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Let C be a nice (smooth, projective, geometrically integral) curve over k of genus g given by equations. Let J be the Jacobian of C.

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Given the equations of C, compute the endomorphism ring $\mathbf{End}\, J^{\mathrm{al}}$.

- It is an interesting challenge.
- If $\mathbf{End}\, J$ contains non-trivial idempotents, we can hope to decompose J into abelian varieties of smaller dimension.
- If $\mathbf{End}\, J$ is non-trivial, then this allows us to find a modular form that describes the arithmetic properties of J and C.
- Can be used to show transcendence of 1-periods (Ouaknine–Worrell–Sertöz)

An analytic description of the Jacobian

Via a chosen embedding of k into $\mathbb{C}$ and a projection into $\mathbb{P}^2$, we can consider C as a Riemann surface, and

$$J_{\mathbb{C}} = H^0(C, \Omega_C)^\vee / H_1(C, \mathbb{Z}) = \mathbb{C}^g / \Lambda,$$

where we pick an k basis for $H^0(C, \Omega_C) = k\omega_1 \oplus \ldots \oplus k\omega_g$, hence,

$$\Lambda = \left\{ \left(\int_\gamma \omega_1, \ldots, \int_\gamma \omega_g \right) \in \mathbb{C}^g \ : \ \gamma \in H_1(C, \mathbb{Z}) \right\} \cong \mathbb{Z}^{2g}.$$

In other words, J is a complex torus (plus a polarization).

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- We can calculate Λ numerically by taking a plane model
- Using Λ, we can hope to understand J analytically...
  and perhaps even be able to transfer these results to the algebraic setting.

Heuristic solution

By picking a k-basis for $H^0(C, \Omega_C)$, we have

$$\mathrm{End}(J) = \{T \in M_g(k) \mid T\Lambda \subset \Lambda\}$$

Hence, if Π is a period matrix for C, i.e., $\Lambda = \Pi\mathbb{Z}^{2g}$, then we are reduced to finding a $\mathbb{Z}$-basis of the solutions (T, R) to

$$T\Pi = \Pi R, \qquad T \in M_g(k^{\mathrm{al}}), \quad R \in M_{2g}(\mathbb{Z}).$$

The Galois module structure of $\mathrm{End}(J^{\mathrm{al}})$ is given via its action on $T \in M_g(k^{\mathrm{al}})$.

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There is no obvious way to prove that our guesses are actually correct.

Representing endomorphisms via correspondences

$$\alpha_C : C \xrightarrow{AJ} J \xrightarrow{\alpha} J \dashrightarrow \mathrm{Sym}^g(C)$$

$$P \mapsto \{Q_1, \ldots, Q_g\} \iff \alpha([P - P_0]) = \left[\sum_{i=1}^{g} Q_i - P_0\right]$$

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Theorem (C–Mascot–Sijsling–Voight)

We give an algorithm for

$$\mathrm{M}_g(k^{\mathrm{al}}) \ni \alpha \mapsto \begin{cases} \texttt{true} & \text{if } \alpha \in \mathrm{End}\, J^{\mathrm{al}}, \text{ and a certificate} \\ \texttt{false} & \text{if } \alpha \notin \mathrm{End}\, J^{\mathrm{al}} \end{cases}$$

By interpolation via α_C or by locally solving a differential equation on $C \times C$.

Rigorous Endomorphism ring

Theorem (C–Mascot–Sijsling–Voight, C–Lombardo–Voight, C–Sertöz)

We give an algorithm that computes $\operatorname{End} J^{\mathrm{al}}$ *with a certificate* .

This is a day/night algorithm:

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$$\{L_{p_1}(t) := \det(1 - t\,\mathrm{Frob}_p | H^1), L_{p_2}(t), \dots, L_{p_i}(t)\} \longmapsto \text{upper bounds on } \operatorname{End} J^{\mathrm{al}}$$

- The $L_p(t)$ polynomials are as random as $\operatorname{End} J^{\mathrm{al}}$ allows it.
- Two polynomials $L_p(t)$ and $L_q(t)$ give an upperbound for $\operatorname{End} J^{\mathrm{al}}$.

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  - Two polynomials $L_p(t)$ and $L_q(t)$ give an upperbound for $\operatorname{End} J^{\mathrm{al}}$.
  - this upper bound is sharp for (p, q) in a set of positive density, under the Mumford–Tate conjecture,
  - The set is unknown apriori.

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$$\operatorname{Frob}_p \bmod p^N \circlearrowright H^1_{\mathrm{crys}}(C, \mathbb{Z}_p) \longmapsto \text{upper bounds on } \operatorname{End} J^{\mathrm{al}}$$

- We check what correspondences $C \rightsquigarrow C \bmod p$ lift to $C \rightsquigarrow C \bmod p^N$.
- $\operatorname{Frob}_p \bmod p^N$ is a byproduct of computing $L_p(t) = \det(1 - t \operatorname{Frob}_p | H^1_{MW})$.

Examples

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- The algorithm verifies that the following genus 4 curve over $\mathbb{Q}(\sqrt{3})$

$$0 = -8x^2 + 8xy + 17y^2 - 34xz - 2yz - 28z^2 - 10xw - 9yw - 18zw + 2w^2,$$

$$\begin{aligned} 0 = 4x^3 - 6x^2y - 6xy^2 + 12x^2z + 6xyz + 24y^2z - 12xz^2 - 24z^3 + 2x^2w + 7xyw \\ + 4y^2w + 4xzw - 13yzw - 8z^2w - 20xw^2 - 3zw^2 - 12w^3 \end{aligned}$$

has real multiplication by the maximal order of $\mathbb{Q}(x)/(x^4 - x^3 - 3x^2 + x + 1)$.

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  The first step to show that, under Langlands, it corresponds to a specific Hilbert modular form f, i.e., $J_{\mathbb{Q}(\sqrt{3})} \sim A_f$.
  We used this in a recent project, where we show that the 2-isogeny field of A_f solves the inverse Galois problem for $\mathrm{PSL}_2(\mathbb{F}_{16}) \rtimes C_2 \simeq \mathbf{17T7}$.

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  We used this in a recent project, where we show that the 2-isogeny field of A_f solves the inverse Galois problem for $\mathrm{PSL}_2(\mathbb{F}_{16}) \rtimes C_2 \simeq$ 17T7.
- Our method works just as well for isogenies and projections.
- Try it: https://github.com/edgarcosta/endomorphisms

What is a K3 surface?

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They may arise in many ways:

- smooth quartic surface in $\mathbb{P}^3$

$$X : f(x, y, z, w) = 0, \quad \deg f = 4$$

e.g. Fermat quartic surface $x^4 + y^4 + z^4 + w^4 = 0$.

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- Kummer surfaces, $\mathrm{Kummer}(A) := \widetilde{A/\pm}$, with A an abelian surface.

Picard lattice of a K3 surface

Let X be a K3 surface defined over $k \subset \mathbb{C}$. We view X also as a complex manifold.

$$\mathrm{NS}\, X^{\mathrm{al}} \simeq \mathrm{Pic}\, X^{\mathrm{al}} \simeq \mathbb{Z}\langle \text{algebraic curves in } X \rangle / \langle \text{linear equivalences} \rangle \subset H_2(X, \mathbb{Z})$$

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From the equations of X, compute $\operatorname{Pic} X^{\mathrm{al}} \subset H_2(X, \mathbb{Z})$ as a $\operatorname{Gal}(k^{\mathrm{al}}/k)$-module.

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Useful for studying rational points, via a potential Brauer–Manin obstruction:

$$H^1(\mathrm{Gal}(k^{\mathrm{al}}/k), \operatorname{Pic} X^{\mathrm{al}}) \simeq \mathrm{Br}_1(X)/Br_0(X)$$

$$X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k)$$

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From the equations of X, compute $\operatorname{Pic} X^{\mathrm{al}} \subset H_2(X, \mathbb{Z})$ as a $\operatorname{Gal}(k^{\mathrm{al}}/k)$-module.

Over a finite field, the Tate conjecture (proven) gives us $\operatorname{rk} \operatorname{Pic} X$ from

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and Artin–Tate conjecture (proven) also gives $\operatorname{disc} \operatorname{Pic} X^{\mathrm{al}}$ modulo squares.

Picard lattice of a K3 surface

Let X be a K3 surface defined over $k \subset \mathbb{C}$. We view X also as a complex manifold.

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Over a number field, there are several in principle algorithms to compute $\operatorname{rk} \operatorname{Pic} X$ or even $\operatorname{Pic} X$. These involve, a day/night algorithm:

- by day: find curve classes in $\operatorname{Pic} X$;
- by night: restrict the ambient space for $\operatorname{Pic} X \subset H^2(X, \mathbb{Z})$.

An analytic approach

Lefschetz (1,1) theorem

A homology class $\gamma \in H_2(X, \mathbb{Z})$ is in $\operatorname{Pic} X^{\mathrm{al}}$ if and only if $\int_\gamma \omega_X = 0$, where ω_X is the nonzero holomorphic 2-form ω_X on X, unique up to scaling.

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Hence, if $\Pi = [\int_\gamma \omega_X]_{\gamma \in H_2(X,\mathbb{Z})} \in \mathbb{C}^{22}$ is the period vector for ω_X, then we are reduced to finding a (saturated) lattice $\Lambda \subset H_2(X, \mathbb{Z})$ of solutions

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- Nonetheless, given Π as a ball, one can compute $B \gg 0$ such that such that

$$\operatorname{Pic}(X^{\mathrm{al}})_{|B} := \mathbb{Z}\langle \gamma \in \operatorname{Pic} X^{\mathrm{al}} \mid -\gamma_{\mathrm{prim}}^2 < B \rangle \subseteq \Lambda \qquad \text{(Lairez–Sertöz).}$$

A running example inspired by Klein–Mukai

$$X : x^4 + xyzw + y^3z + yw^3 + z^3w = 0 \subset \mathbb{P}^3$$

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- Lattice computations with Λ predict that there are

  133056

  smooth rational quartics spanning Λ.

Reconstructing isolated curves from their Hodge classes

Turns out one can compute a bit more for hypersurfaces

$$\varphi \colon H_2(X,\mathbb{Z}) \times H^2_{\mathrm{dR}}(X/k) \to \mathbb{C} \qquad (\gamma,\omega) \longmapsto \int_\gamma \omega$$

Note, if $\gamma \in \operatorname{Pic} X^{\mathrm{al}}$, then $\frac{1}{2\pi i}\int_\gamma \omega \in k^{\mathrm{al}}$ for $\omega \in F^1 H^2_{\mathrm{dR}}(X/k)$.

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Theorem (Movasati–Sertöz)

If $\gamma = [C] \in H_2(X,\mathbb{Z})$ for a curve $C \subset X$ then from $\frac{1}{2\pi i}(\int_\gamma \omega)_{\omega \in F^1}$ one can construct an ideal I_γ such that $I(C) \subsetneq I_\gamma$.

In favorable circumstances we expect low order equations in I_γ to span $I(C)$.

For example, smooth rational curves of degree up to 4 in K3s.

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Theorem (Cifani–Pirola–Schlesinger)

For a smooth rational quartic curve $C \subset X$ we have that the equation of the quadric surface containing C generates $I_{[C],2}$, i.e., $I(C)_2 = I_{[C],2}$.

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- Fortunately, there is a small $\mathbf{Aut}(\Lambda)$ orbit of size 336:
  $133056 = 336 + 1008 + 1176 + 3528 \cdot 3 + 4704 \cdot 3 + 7056 \cdot 9 + 14112 \cdot 3$
- For each quartic curve $C \subset X$, we can compute

$$I_{[C],2} = \langle a_0 x^2 + \cdots + a_9 w^2 \rangle_{\mathbb{C}}$$

  that defines a quadric surface Q, such that $Q \cap X = C \cup \overline{C}$.
  Hence, we expect an orbit of 168 quadrics each containing a pair of quartics.
- We aim reconstruct the ten (algebraic!) coefficients of these quadrics.

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- The minimal polynomials have large height about 9k characters, e.g.:

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  The abstract isomorphism problem feels hopeless. 😨

Isomorphism problem

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Construct $\mathbb{Q}(a_k) \hookrightarrow L$, where $L = \mathbb{Q}(a_0, \ldots, a_9) = \mathbb{Q}(a_0)$.

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In our case, we have all the compatible embeddings

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Thus the isomorphisms is given is the solution of the following linear system

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In practice, it is faster to iteratively refine the complex embeddings, as their height is smaller than theoretically possible: 4k vs 120k digits.

Intersecting the quadric surfaces with the K3 surface

$$Q : a_0x^2 + a_1xy + \cdots + a_9w^2 = 0 \subset \mathbb{P}^3, \quad [L := \mathbb{Q}(\{a_i\}_i) : \mathbb{Q}] = 168$$

Goal

Show that $Q \cap X$ decomposes into two quartic curves.

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- We conclude $\deg S = 10$ via Gotzmann regularity theorem, by checking that $\dim L[x, y, z, w]_\bullet / I_\bullet = 10$ for $\bullet = 6, 7$, where $V(I) = S$.

Certifying $\operatorname{Pic} X^{\mathrm{al}} = \Lambda$

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$Q \cap X$ decomposes into a pair of quartics over K a quadratic extension of L.

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Compute K and $\mathrm{Gal}(K/\mathbb{Q})$ acting on Λ_Q.

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Can one compute K using geometry without Gröbner basis?

To try: For a generic hyperplane $Q \cap X \cap H$ is a degree 8 reduced scheme.
The number field K is the quadratic extension where we observe two orbits.

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Can we compute $\mathrm{Gal}(K/\mathbb{Q})$? $\mathrm{Gal}(K/\mathbb{Q}) \overset{?}{=} \mathrm{Aut}\,\Lambda$? $H^1(\mathrm{Gal}(k^{\mathrm{al}}/k), \mathrm{Pic}\,X^{\mathrm{al}}) =?$

Summary

Theorem (C–Sertöz)

The quartic surface $X : x^4 + xyzw + y^3z + yw^3 + z^3w = 0 \subset \mathbb{P}^3$ has $\operatorname{Pic} X^{\mathrm{al}} = \Lambda$, generated by quartics over a quadratic extension of $L := \mathbb{Q}(\{a_i\}_i)$.

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Do you have a challenge K3 surface for us?