Effective Computation of Picard lattices of K3 surfaces

Edgar Costa (MIT)

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Slides available at edgarcosta.org. Joint work with Emre Can Sertöz

“Dans la seconde partie de mon rapport, il s’agit des variétés kähleriennes dites K3, ainsi nommées en l’honneur de Kummer, Kähler, Kodaira et de la belle montagne K2 au Cachemire.” —André Weil *(Photo credit: Waqas Anees)*

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- smooth quartic surface in $\mathbb{P}^3$

$$X : f(x, y, z, w) = 0, \quad \deg f = 4$$

  e.g. Fermat quartic surface $x^4 + y^4 + z^4 + w^4 = 0$.

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- double cover of $\mathbb{P}^2$ branched over a sextic curve $\mathbb{P}(3, 1, 1, 1)$

$$X : w^2 = f(x, y, z), \quad \deg f = 6$$

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- Kummer surfaces, $\mathrm{Kummer}(A) := \widetilde{A/\pm}$, with A an abelian surface.

Picard lattice of a K3 surface

Let X be a K3 surface defined over $k \subset \mathbb{C}$. We view X also as a complex manifold.

$$\mathrm{NS}(X^{\mathrm{al}}) \simeq \mathrm{Pic}\, X^{\mathrm{al}} \simeq \mathbb{Z}\langle \text{algebraic curves in } X \rangle / \langle \text{linear equivalences} \rangle \subset H_2(X, \mathbb{Z})$$

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$$\mathrm{Pic}(X^{\mathrm{al}}) \simeq H^{1,1}(X) \cap H^2(X, \mathbb{Z}) \subsetneq H^2(X, \mathbb{Z}) \simeq (-E_8)^2 \oplus U^3 \simeq \mathbb{Z}^{22}$$

and $\mathrm{rk}\,\mathrm{Pic}\, X^{\mathrm{al}} \in \{1, 2, \ldots, 20\}$. For a generic K3 surface we have $\mathrm{rk}\,\mathrm{Pic}\, X^{\mathrm{al}} = 1$

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“New and interesting” Galois representations arise from $T(X)$.

$$H^2(X, \mathbb{Q}) \simeq Pic(X^{\mathrm{al}})_{\mathbb{Q}} \oplus T(X)_{\mathbb{Q}}$$

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Goal

From the equations of X, compute $\operatorname{Pic} X^{\mathrm{al}} \subset H_2(X, \mathbb{Z})$ as a $\operatorname{Gal}(k^{\mathrm{al}}/k)$-module.

"The evaluation of ρ for a given surface presents in general grave difficulties." — Zariski

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- Over a number field there are several in principle algorithms to compute $\operatorname{rk} \operatorname{Pic} X$ or even $\operatorname{Pic} X$. These involve, a day/night algorithm:
  - by day: find curve classes in $\operatorname{Pic} X$;
  - by night: restrict the ambient space for $\operatorname{Pic} X \subset H^2(X, \mathbb{Z})$.

An analytic approach

Lefschetz (1,1) theorem

A homology class $\gamma \in H_2(X, \mathbb{Z})$ is in $\operatorname{Pic} X^{\mathrm{al}}$ if and only if $\int_\gamma \omega_X = 0$, where ω_X is the nonzero holomorphic 2-form ω_X on X, unique up to scaling.

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Hence, if $\Pi \in \mathbb{C}^{22}$ is the period vector for ω_X, i.e., $[\int_\gamma \omega_X]_{\gamma \in H_2(X,\mathbb{Z})}$, then we are reduced to finding a lattice $\Lambda \subset H_2(X, \mathbb{Z})$ of solutions

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- There is no obvious way to prove that our guesses are actually correct.
- Nonetheless, a posteriori, one can compute $B \gg 0$ such that

$$\operatorname{Pic}(X^{\mathrm{al}})_{|B} := \mathbb{Z}\langle \gamma \in \operatorname{Pic} X^{\mathrm{al}} \mid -\gamma_{\mathrm{prim}}^2 < B \rangle \subseteq \Lambda \qquad \text{(Lairez–Sertöz).}$$

A running example inspired by Klein–Mukai

$$X : x^4 + xyzw + y^3z + yw^3 + z^3w = 0 \subset \mathbb{P}^3$$

- It is a fiber in a pencil that has generic rank 19 and matching upper bounds can be deduced by positive characteristic methods.

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  - The Picard rank of a K3 surface over $\mathbb{F}_p^{\mathrm{al}}$ is always even.
  - If p and q are two primes of good reduction, such that

    $$\operatorname{rk}\operatorname{Pic} X_{\mathbb{F}_p}^{\mathrm{al}} = \operatorname{rk}\operatorname{Pic} X_{\mathbb{F}_q}^{\mathrm{al}} = 20 \text{ and } \operatorname{disc}\operatorname{Pic} X_{\mathbb{F}_p}^{\mathrm{al}} \neq \operatorname{disc}\operatorname{Pic} X_{\mathbb{F}_q}^{\mathrm{al}} \implies \operatorname{rk}\operatorname{Pic} X^{\mathrm{al}} < 20.$$

    this can be deduced from $\det(1 - t\,\mathrm{Frob}_\bullet | H^2(X, \mathbb{Q}_\ell))$. (van Luijk)
  - Use p-adic variational Hodge conjecture to show that only at most 19 classes lift to characteristic zero. One needs $\mathrm{Frob}_p \bmod p^N \circlearrowright H^1_{\mathrm{crys}}(C, \mathbb{Z}_p)$ (C–Sertöz)

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- Heuristically, one computes $\Lambda \simeq \mathbb{Z}^{19}$ such that

$$\Pi\Lambda \approx 0 \qquad \operatorname{Pic}(X^{\mathrm{al}})_{|B} \subseteq \Lambda \overset{?}{\subseteq} \operatorname{Pic} X^{\mathrm{al}}.$$

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- The “smallest” non-trivial curves that appear are smooth rational quartics.
- Lattice computations with Λ predict that there are

$$133056$$

smooth rational quartics spanning Λ.

Reconstructing isolated curves from their Hodge classes

Turns out one can compute a bit more for hypersurfaces

$$\varphi\colon H_2(X,\mathbb{Z}) \times H^2_{\mathrm{dR}}(X/k) \to \mathbb{C} \qquad (\gamma,\omega) \longmapsto \int_\gamma \omega$$

Note, if $\gamma \in \operatorname{Pic} X^{\mathrm{al}}$, then $\frac{1}{2\pi i}\int_\gamma \omega \in k^{\mathrm{al}}$ for $\omega \in F^1 H^2_{\mathrm{dR}}(X/k)$.

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Theorem (Movasati–Sertöz)

If $\gamma = [Y] \in H_2(X,\mathbb{Z})$ for a curve $Y \subset X$ then from $\frac{1}{2\pi i}(\int_\gamma \omega)_{\omega \in F^1}$ one can construct an ideal I_γ such that $I(Y) \subsetneq I_\gamma$.

In favorable circumstances we expect low order equations in I_γ to span $I(Y)$.

For example, smooth rational curves of degree up to 4 in K3s.

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Theorem (Cifani–Pirola–Schlesinger)

For a smooth rational quartic $Y \subset X$ we have that the equation of the quadric containing Y generates $I_{[Y],2}$, i.e., $I(Y)_2 = I_{[Y],2}$.

Reconstructing quadric equations

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- Hence, we expect an orbit of 168 quadrics each containing a pair of quartics.
- We aim reconstruct the ten (algebraic!) coefficients of these quadrics.

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- The minimal polynomials have large height (∼9k characters), e.g.:

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- We solve the isomorphism problem by refining the (matched) complex embeddings and then reconstructing the isomorphism by inverting a (numerically stable) Vandermonde matrix. The isomorphisms have even larger height. (100k characters)

Intersecting the quadric with the K3 surface

$$Q : a_0x^2 + a_1xy + \cdots + a_9w^2 = 0 \subset \mathbb{P}^3, \quad [L := \mathbb{Q}(\{a_i\}_i) : \mathbb{Q}] = 168$$

Goal

Show that $Q \cap X$ decomposes into two quartic curves.

- It suffices to show that the singular locus S of $Q \cap X$ consists of 10 distinct reduced points.

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- Working over $\mathbb{F}_p$ we find 10 distinct points.
  Hence, S is zero-dimensional and reduced, and $\deg S \leq 10$.

Intersecting the quadric with the K3 surface

$$Q : a_0x^2 + a_1xy + \cdots + a_9w^2 = 0 \subset \mathbb{P}^3, \quad [L := \mathbb{Q}(\{a_i\}_i) : \mathbb{Q}] = 168$$

Goal

Show that $Q \cap X$ decomposes into two quartic curves.

- It suffices to show that the singular locus S of $Q \cap X$ consists of 10 distinct reduced points.
- Hopeless to do this directly! Operations in L are seriously expensive!
- Working over $\mathbb{F}_p$ we find 10 distinct points.
  Hence, S is zero-dimensional and reduced, and $\deg S \leq 10$.
- We conclude $\deg S = 10$ via Gotzmann regularity theorem, by checking that $\dim L[x, y, z, w]_\bullet / I_\bullet = 10$ for $\bullet = 6, 7$, where $V(I) = S$.

Certifying $\operatorname{Pic} X^{\mathrm{al}} = \Lambda$

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The inclusion $\Lambda_Q \subseteq \Lambda$ is not explicit!

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- Compute the intersections of these 336 curves with each other over $\mathbb{F}_p$.
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  Showing that there are at most 66528 distinct quadrics. Can be done over $\mathbb{C}$.
  This establishes a bijection between these quadrics and the 168 pairs of quartic curve classes that they correspond to.

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$Q \cap X$ decomposes into a pair of quartics over K a quadratic extension of L.

Goal

Compute K and $\mathrm{Gal}(K/\mathbb{Q})$ acting on Λ_Q.

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Can one compute K using geometry without Gröbner basis?

To try: For a generic hyperplane $Q \cap X \cap H$ is a degree 8 reduced scheme.
The number field K is the quadratic extension where we observe two orbits.

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We guess that $K = F(\sqrt[14]{u})$ for a unit u of where F is defined by

$$x^{24} + x^{22} - 24x^{21} - 84x^{20} - 205x^{19} - 155x^{18} - 770x^{17} - 500x^{16} + 18916x^{15} + 36988x^{14} + 109234x^{13} + 387901x^{12} + 373961x^{11}$$
$$- 18170x^{10} + 75132x^9 + 10381x^8 - 123071x^7 + 108274x^6 - 41580x^5 + 39936x^4 - 21911x^3 + 4032x^2 + 1428x + 616$$

and $\mathrm{Gal}(F/\mathbb{Q}) = C_3 \times \mathrm{PGL}(2,7)$. $\#\,\mathrm{Gal}(F/\mathbb{Q})$ is 14 times smaller than $\#\,\mathrm{Aut}\,\mathrm{Pic}\,X^{\mathrm{al}}$.

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Can we compute $\mathsf{Gal}(K/\mathbb{Q})$ by hand? $\mathsf{Gal}(K/\mathbb{Q}) \stackrel{?}{=} \mathsf{Aut}\,\Lambda$? $H^1(\mathsf{Gal}(k^{\mathrm{al}}/k), \mathsf{Pic}\,X^{\mathrm{al}}) =?$

Summary

Theorem (C–Sertöz)

The quartic surface $X : x^4 + xyzw + y^3z + yw^3 + z^3w = 0 \subset \mathbb{P}^3$ has $\operatorname{Pic} X^{\mathrm{al}} = \Lambda$, generated by quartics over a quadratic extension of $L := \mathbb{Q}(\{a_i\}_i)$.

We are hoping to streamline this method and also figure out its applications/limitations.

Hopefully, also be able handle families, e.g.,

$$X : x^4 + \lambda xyzw + y^3z + yw^3 + z^3w = 0 \subset \mathbb{P}^3(\mathbb{Q}(\lambda))$$

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Do you have a challenge K3 surface for us?