Computing zeta functions of nondegenerate hypersurfaces in toric varieties

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Joint work with David Harvey (UNSW) and Kiran Kedlaya (UCSD)

Slides available at `edgarcosta.org` under Research

The zeta function problem

- $\mathbb{F}_q$ finite field of characteristic p
- X a smooth variety over $\mathbb{F}_q$

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$$\zeta_X(t) := exp\left(\sum_{i\geq 1} \#X(\mathbb{F}_{q^i})\frac{t^i}{i}\right) \in \mathbb{Q}(t)$$

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- In practice, this only works for very few classes of varieties
- Some applications include:
  - L-functions and their special values
  - $\mathrm{End}(A)$ for an abelian variety
  - Arithmetic statistics (Sato–Tate, Lang–Trotter, etc)
  - Other geometric invariants

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A quasi-linear in p algorithm for hypersurfaces in toric varieties.

Hypersurfaces in toric varieties

Toy example, the Projective space

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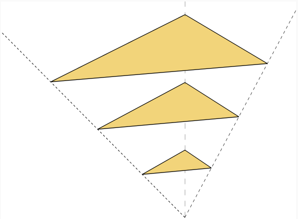

- We can think of $P_d := R[d\Delta \cap \mathbb{Z}^n]$, where Δ is the standard simplex.
- Idea: generalize Δ to be any polytope.

Toric hypersurfaces

- $f = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha x^\alpha \in R[x_1^\pm, \dots, x_n^\pm]$ a Laurent polynomial
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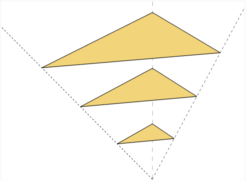
$$P_\Delta := \bigoplus_{d \geq 0} P_d, \quad P_d := R[x^\alpha : \alpha \in d\Delta \cap \mathbb{Z}^n]$$

$$\mathbb{P}_\Delta := \mathrm{Proj}\, P_\Delta$$

$$X_f := \mathrm{Proj}\, P_\Delta/(f) \subset \mathbb{P}_\Delta$$



X_f is an hypersurface in the toric variety $\mathbb{P}_\Delta$

Toric hypersurfaces are everywhere

Vertices of Δ	Resulting hypersurface
$0, e_1, \ldots, e_n$	Hypersurface in $\mathbb{P}^n$
$0, (2g+1)e_1, 2e_2$	Odd hyperelliptic curve of genus g
$0, ae_1, be_2$	$C_{a,b}$-curve
$0, 4e_1, 4e_2, 4e_3$	Quartic K3 surface
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- in $\mathbb{P}^3$, as a quartic surface;
- in 95 weighed projective spaces (Reid's list);
- in **4319** toric varieties.

Keeping our eyes on the prize

Given

$$f = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha x^\alpha \in \mathbb{F}_q[x_1^{\pm}, \ldots, x_n^{\pm}]$$

efficiently compute

$$\begin{aligned} \zeta_X(t) &:= exp\left(\sum_{i \geq 1} \#X(\mathbb{F}_{q^i}) \frac{t^i}{i} \right) \\ &= \det(1 - q^{-1} t \operatorname{Frob} | PH^{\dagger, n-1}(X))^{(-1)^n} \zeta_{\mathbb{P}_\Delta}(t), \end{aligned}$$

where $X := \operatorname{Proj} P_\Delta / (f) \subset \mathbb{P}_\Delta$.

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A generic condition over an infinite field and a fixed Δ

p-adic Cohomology

Master plan

Setup

- $f = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha x^\alpha \in \mathbb{F}_q[x_1^\pm, \ldots, x_n^\pm]$
- $X := \operatorname{Proj} P_\Delta/(f) \subset \mathbb{P}_\Delta$ a nondegenerate hypersurface
- $\sigma := p$-th power Frobenius map

Goal

Compute the matrix representing the action of σ in $PH^{\dagger, n-1}(X)$ with enough p-adic precision to deduce

$$Q(t) = \det(1 - q^{-1} t \operatorname{Frob} | PH^{\dagger, n-1}(X)) \in 1 + \mathbb{Z}[t].$$

We will use **Abbott–Kedlaya–Roe** type algorithm, an adaptation of Kedlaya's algorithm to smooth projective hypersurfaces.

Overall picture for an Abbott–Kedlaya–Roe type algorithm

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$PH^{\dagger,n-1}(X)$ ⇢ de Rham cohomology with overconvergent power series

cohomology relations + commutative algebra $\Longrightarrow$ basis for $PH^{n-1}_{\mathrm{dR}}(X_{\mathbb{Q}_q})$ + reduction algorithm

Examples

Example: K3 surface in the Dwork pencil

X a projective quartic surface in $\mathbb{P}^3_{\mathbb{F}_p}$ defined by

$$x^4 + y^4 + z^4 + w^4 + \lambda xyzw = 0.$$

For $\lambda = 1$ and $p = 2^{20} - 3$, using the old projective code in `22h7m` we compute that

$$\zeta_X(t)^{-1} = (1-t)(1-pt)^{16}(1+pt)^3(1-p^2t)Q(t),$$

where the "interesting" factor is

$$Q(t) = (1+pt)(1 - 1688538t + p^2t^2).$$

Example: a quartic surface in the Dwork pencil

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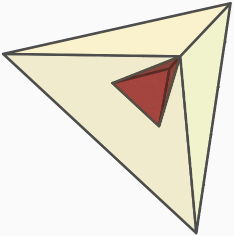
The defining monomials of X generate a sublattice of index 4^2 in $\mathbb{Z}^3$, and we can work “in” that sublattice, by using

$$x^4y^{-1}z^{-1} + \lambda x + y + z + 1 = 0$$

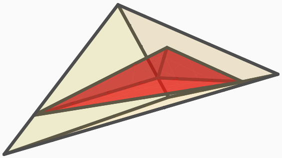
which has a polytope much smaller than the full simplex (32/3 `vs` 2/3).



Example: a hypergeometric motive (also a K3 surface)

Consider the appropriate completion of the toric surface over $\mathbb{F}_p$ with $p = 2^{15} - 19$ given by

$$x^3y + y^4 + z^4 - 12xyz + 1 = 0.$$



In `4s`, we compute that the "interesting" factor of $\zeta_X(t)$ is (up to rescaling)

$$pQ(t/p) = p + 20508t^1 - 18468t^2 - 26378t^3 - 18468t^4 + 20508t^5 + pt^6.$$

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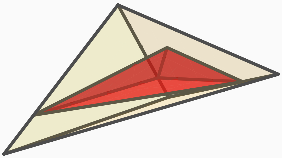

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We can confirm the linear term with Magma:

```
C2F2 := HypergeometricData([6,12], [1,1,1,2,3]);
EulerFactor(C2F2, 2^10 * 3^6, 2^15-19: Degree:=1);
1 + 20508*$.1 + O($.1^2)
```

Example: a K3 surface in a non weighted projective space

Consider the surface X defined as the closure (in $\mathbb{P}_\Delta$) of the affine surface defined by the Laurent polynomial

$$3x + y + z + x^{-2}y^2z + x^3y^{-6}z^{-2} + 3x^{-2}y^{-1}z^{-2} \\ - 2 - x^{-1}y - y^{-1}z^{-1} - x^2y^{-4}z^{-1} - xy^{-3}z^{-1}.$$

The Hodge numbers of $PH^2(X)$ are $(1, 14, 1)$. For $p = 2^{15} - 19$, in `6m20s` we obtain the "interesting" factor of $\zeta_X(t)$:

$$pQ(t/p) = (1-t)\cdot(1+t)\cdot(p + 33305t^1 + 1564t^2 - 14296t^3 - 11865t^4 \\ + 5107t^5 + 27955t^6 + 25963t^7 + 27955t^8 + 5107t^9 \\ - 11865t^{10} - 14296t^{11} + 1564t^{12} + 33305t^{13} + pt^{14}).$$

We know of no previous algorithm that can compute $\zeta_X(t)$ for p in this range!

Example: a quintic threefold in the Dwork pencil

Consider the threefold X in $\mathbb{P}^4_{\mathbb{F}_p}$ for $p = 2^{20} - 3$ given by

$$x_0^5 + \cdots + x_4^5 + x_0 x_1 x_2 x_3 x_5 = 0.$$

In `11m18s`, we compute that

$$\zeta_X(t) = \frac{R_1(pt)^{20} R_2(pt)^{30} S(t)}{(1-t)(1-pt)(1-p^2t)(1-p^3t)}$$

where the "interesting" factor is

$$S(t) = 1 + 74132440T + 748796652370pT^2 + 74132440p^3T^3 + p^6T^4.$$

and R_1 and R_2 are the numerators of the zeta functions of certain curves (given by a formula of Candelas–de la Ossa–Rodriguez Villegas).

Using the old projective code, we extrapolate it would have taken us at least 120 days.

Example: a Calabi–Yau 3fold in a non weighted projective space

Let X be the closure (in $\mathbb{P}_\Delta$) of the affine threefold

$$xyz^2w^3 + x + y + z - 1 + y^{-1}z^{-1} + x^{-2}y^{-1}z^{-2}w^{-3} = 0.$$

For $p = 2^{20} - 3$, in `1h15m`, we computed the "interesting" factor of $\zeta_X(t)$

$$(1+718pt+p^3t^2)(1+1188466826t+1915150034310pt^2+1188466826p^3t^3+p^6t^4).$$

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By analogy with Reid's list, Calabi–Yau threefolds can arise as hypersurfaces in:

- 7555 weighted projective spaces;
- 473,800,776 toric varieties.

See `http://hep.itp.tuwien.ac.at/~kreuzer/CY/`.

Example: a cubic fourfold

X a cubic fourfold in $\mathbb{P}^5$ defined by the zero locus of

$$x_0^3+x_1^3+x_2^3+(x_0+x_1+2x_2)^3+x_3^3+x_4^3+x_5^3+2(x_0+x_3)^3+3(x_1+x_4)^3+(x_2+x_5)^3$$

For $p=31$, in `21h31m` we computed the "interesting" factor of $\zeta_X(t)$

$$\begin{aligned} pQ(t/p^2) = {} & p-7t^1+21t^2-52t^3-8t^4-28t^5+21t^6+35t^7+39t^9+62t^{10}+23t^{11} \\ & +62t^{12}+39t^{13}+35t^{15}+21t^{16}-28t^{17}-8t^{18}-52t^{19}+21t^{20}-7t^{21}+pt^{22} \end{aligned}$$

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For $p=127$ the running time was `23h15m` and for $p=499$ it was `24h55m`.

In both cases, we also observed that the "interesting" factor is an irreducible Weil polynomial.

Most of the time is spent setting up and solving the initial linear algebra problems.