Computing zeta functions of nondegenerate toric hypersurfaces

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Joint work with David Harvey (UNSW) and Kiran Kedlaya (UCSD)

Slides available at `edgarcosta.org` under Research

Motivation

Riemann zeta function

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} \cdots$$
$$= \frac{1}{1-2^{-s}} \cdot \frac{1}{1-3^{-s}} \cdot \frac{1}{1-5^{-s}} \cdots$$

- One of the most famous examples of a global zeta function
- Together with the functional equation

$$\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s) = \xi(1-s)$$

  encodes a lot of the arithmetic information of $\mathbb{Z}$.
  e.g.: Zeros of $\zeta(s) \rightsquigarrow$ precise prime distribution
- $\zeta(s)$ still keeps secret many of its properties

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Hasse and Weil generalized an analog of $\zeta(s)$ for algebraic varieties

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- What arithmetic properties of X can we read from $\zeta_{X_p}(s)$?
- $\zeta_{X_p}(t)$ obeys a functional equation and satisfies the Riemann hypothesis!
- What about $\zeta_X(s)$?

Elliptic curves

E an elliptic curve over $\mathbb{Q}$

$$\zeta_E(s) := \prod_p \zeta_{E_p}(p^{-s}) \quad \text{and} \quad \zeta_{E_p}(t) = \frac{L_p(t)}{(1-t)(1-pt)}$$

$$L_p(t) = \begin{cases} 1 - a_p t + p t^2, & \text{good reduction}, a_p = p + 1 - \#E_p(\mathbb{F}_p) \\ 1 \pm t, & \text{non-split/split multiplicative reduction;} \\ 1 & \text{additive reduction;} \end{cases}$$

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- $a_p \rightsquigarrow$ arithmetic information about $E_p \rightsquigarrow E$.
- Modularity theorem $\implies L_E$ satisfies a functional equation
- Birch–Swinnerton-Dyer conjecture predicts $\mathrm{ord}_{s=1} L_E(s) = \mathrm{rk}(E)$.

$\zeta(s)$ vs $\zeta_X(s)$

We always expect $\zeta_X(s)$ to satisfy a functional equation.

- zero-dimensional varieties (number fields) ✓
- elliptic curves over $\mathbb{Q}$ ✓
- genus 2 curves ?

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- easy to explicitly write down $\zeta(s)$
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Problem

Given an *explicit* description of X, compute

$$\zeta_{X_p}(t) := exp\left(\sum_{i\geq 0} \#X_p(\mathbb{F}_{p^i})\frac{t^i}{i}\right) \in \mathbb{Q}(t)$$

The zeta function problem

Let X be a smooth variety over a finite field $\mathbb{F}_q$ of characteristic p, consider

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Compute ζ_X from an *explicit* description of X.

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This approach is only practical for very few classes of varieties, e.g., low genus curves and p small.

Real life applications

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- Arithmetic statistics
  - Sato–Tate
  - Lang–Trotter

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- p-adic: based on Monsky–Washnitzer cohomology

Today

New p-adic method to compute $\zeta_X(t)$ that achieves a striking balance between **practicality** and **generality**.

Outline

Toric hypersurfaces

p-adic Cohomology

Some examples

Toric hypersurfaces

Toy example, the Projective space

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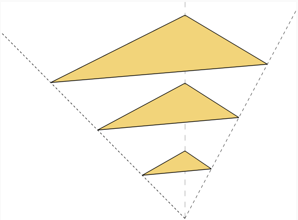
- We can think of $P_d := R[d\Delta \cap \mathbb{Z}^n]$, where Δ is the standard simplex.
- Idea: generalize Δ to be any polytope.



Toric hypersurfaces

- $f = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha x^\alpha \in R[x_1^\pm, \ldots, x_n^\pm]$ a Laurent polynomial
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$$P_\Delta := \bigoplus_{d \geq 0} P_d, \quad P_d := R[d\Delta \cap \mathbb{Z}^n]$$

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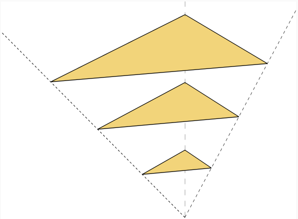
$$X_f := \mathrm{Proj}\, P_\Delta/(f) \subset \mathbb{P}_\Delta$$



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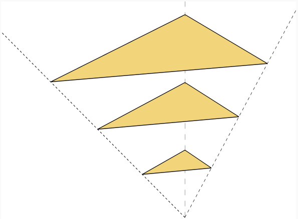

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Examples

Δ	X_Δ
$\mathrm{Conv}(0, e_1, \dots, e_n)$	$\mathbb{P}^n$
$\mathrm{Conv}(0, e_1, \ell e_2, \dots, \ell e_n)$	$\mathbb{P}^n(\ell, 1, \dots, 1)$
$\mathrm{Conv}(0, e_1, e_2, e_1 + e_2) = [0, 1]^2$	$\mathbb{P}^1 \times \mathbb{P}^1$

Toric hypersurfaces are everywhere

Some familiar examples:

Vertices of Δ	Resulting hypersurface
$0, de_1, de_2$	Smooth plane curve of genus $\binom{d-1}{2}$
$0, (2g+1)e_1, 2e_2$	Odd hyperelliptic curve of genus g
$0, ae_1, be_2$	$C_{a,b}$-curve
$0, 4e_1, 4e_2, 4e_3$	Quartic K3 surface
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K3 surfaces can arise as hypersurfaces:

- in $\mathbb{P}^3$, as a quartic surface;
- in 95 weighed projective spaces;
- in 4319 toric varieties.

Keeping our eyes on the prize

Given

$$f = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha x^\alpha \in \mathbb{F}_q[x_1^{\pm}, \dots, x_n^{\pm}]$$

efficiently compute

$$\zeta_X(t) := exp\left(\sum_{i \geq 1} \#X(\mathbb{F}_{q^i}) \frac{t^i}{i} \right)$$
$$= \prod_i \det\left(1 - t\,\mathrm{Frob}\,|H^i_{\mathrm{et}}(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell)\right)^{(-1)^{i+1}} \in \mathbb{Q}(t),$$

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We will need a bit more, we will **nondegeneracy**.

Nondegenerate toric hypersurfaces

Geometric definition

An hypersurface is **nondegenerate** if the cross-section by any bounding hyperplane (in any dimension) are all smooth in their respective tori.

Equivalently, if for every face $\sigma \subseteq \Delta$, f restricted to the torus associated to σ is nonsingular of codimension 1.

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Example

Let C be a plane curve in $\mathbb{P}^2$, then C is nondegenerate if:

- C does not pass through the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$;
- C intersects the coordinate axes $x = 0$, $y = 0$, $z = 0$ transversally;
- C is smooth on the complement of the coordinate axes.

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In terms of ideals, rad $\left\langle x\frac{\partial}{\partial x}f, y\frac{\partial}{\partial y}f, z\frac{\partial}{\partial z}f, f\right\rangle = \langle x, y, z\rangle$

p-adic Cohomology

Goal

Setup

- $f = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha x^\alpha \in \mathbb{F}_q[x_1^\pm, \dots, x_n^\pm]$
- $X := \operatorname{Proj} P_\Delta/(f) \subset \mathbb{P}_\Delta$ a nondegenerate hypersurface

Goal

Compute

$$\begin{aligned}
\zeta_X(t) &:= exp\left(\sum_{i \geq 1} \#X(\mathbb{F}_{q^i}) t^i / i\right) \in \mathbb{Q}(t) \\
&= \prod_i \det\left(1 - t \operatorname{Frob} | H^i_{\mathrm{et}}(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell)\right)^{(-1)^{i+1}} \\
&= Q(t)^{(-1)^n} \prod_{i=0}^{n-1} \left(\frac{1}{1 - q^i t}\right)^{b_i},
\end{aligned}$$

where $Q(t) \in 1 + \mathbb{Z}[t]$ and the b_i are determined by Δ.

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In general, to recover $Q(t)$ we need $\#X(\mathbb{F}_{q^a})$ for $a = 1, \ldots, \lceil \deg Q/2 \rceil$.

Some data points:

- Elliptic curve, $Q(t) = 1 - a_p t + p t^2$
- Smooth plane curve, $Q(t) = 1 + a_{p,1} t + a_{p,2} t^2 + p a_{p,1} t^3 + p^2 t^4$
- K3 surface, $\deg Q \leq 21$
- Calabi-Yau 3fold, $\deg Q \leq 204$

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Can we rewrite $\#X(\mathbb{F}_q) = \{x \in \mathbb{F}_q : f(x) = 0\}$?

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In general, to recover $Q(t)$ we need $\#X(\mathbb{F}_{q^a})$ for $a = 1, \ldots, \lceil \deg Q/2 \rceil$.

Some data points:

- Elliptic curve, $Q(t) = 1 - a_p t + pt^2$
- Smooth plane curve, $Q(t) = 1 + a_{p,1}t + a_{p,2}t^2 + pa_{p,1}t^3 + p^2t^4$
- K3 surface, $\deg Q \leq 21$
- Calabi-Yau 3fold, $\deg Q \leq 204$

Can we rewrite $\#X(\mathbb{F}_q) = \{x \in \mathbb{F}_q : f(x) = 0\}$?

$$\cdots = \{x \in X\left(\overline{\mathbb{F}_q}\right) : \mathrm{Frob}_{\mathbb{F}_q}(x) = x\}$$

Lefschetz fixed point theorem

Given

- Z be a nice space;
- H^* be a nice cohomology theory;
- $F : X \to X$ be a nice map.
- $L(F^n) = \#\{x \in X : F^n(x) = x\}$

we have

$$exp\left(\sum_{n=1}^{\infty} L(F^n)t^n/n\right) = \prod_i \det(1 - tF^*|H^i(X))^{(-1)^{i+1}}$$

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$$= \det(1 - q^{-1}t\,\mathrm{Frob}\,|H^n(U_{\overline{\mathbb{F}_q}}))^{(-1)^n} \prod_{i=0}^{n-1}\left(\frac{1}{1-q^i t}\right)^{b_i},$$

where $U := \mathbb{P}_\Delta \backslash X$

Master plan

Setup

- $f = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha x^\alpha \in \mathbb{F}_q[x_1^\pm, \ldots, x_n^\pm]$
- $X := \operatorname{Proj} P_\Delta/(f) \subset \mathbb{P}_\Delta$ a nondegenerate hypersurface
- $U := \mathbb{P}_\Delta \backslash X$ the complement of X
- σ := p-th power Frobenius

Goal

Compute the matrix representing the action of σ in $H^n(U)$ with enough of p-adic precision to deduce

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We will work with $H^n = H^{\dagger, n}$, the Monsky–Washnitzer cohomology.

Overall picture

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$$\Big\downarrow$$

explicit description over $\mathbb{C}$
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$H^{\dagger,n}(U)$ ⇢ de Rham cohomology with overconvergent power series

cohomology relations + commutative algebra $\Longrightarrow$ basis for $H^n_{\mathrm{dR}}(U_{\mathbb{Q}_q}) = \left\{x^\beta \omega / f^i\right\}_\beta$ + reduction algorithm

Generic algorithm – Abbott–Kedlaya–Roe type

$$H^n_{\mathrm{dR}}(U_{\mathbb{Q}_q}) \xrightarrow[\mathrm{id}]{\sim} H^{\dagger,n}(U) \circlearrowright \sigma$$

1. Compute $\left\{ \frac{x^\beta}{f^m} \omega \right\}_\beta$ a monomial basis for $H^n_{\mathrm{dR}}(U_{\mathbb{Q}_q})$ with $\omega := \frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_n}{x_n} \in \Omega^n$
2. In $H^{\dagger,n}$ compute a series approximation for

$$\sigma\left(\frac{x^\beta}{f^m}\omega\right) = p^n \frac{x^{p\beta}}{f^{pm}} \omega \sum_{i \geq 0} \binom{-m}{i} \left(\frac{\sigma(f) - f^p}{f^p}\right)^i$$

3. Write the approximation in terms of basis elements, i.e., apply the de Rham relations

Note: Originally for smooth hypersurfaces in the projective space.

Schematically

Abbott–Kedlaya–Roe	vs	C.–Harvey–Kedlaya
$\sum_{i=0}^{K-1} \binom{-m}{i} \left(\frac{\sigma(f) - f^p}{f^p} \right)^i$		$\sum_{i=0}^{K-1} \binom{-m}{i} \binom{m+K-1}{K-i-1} \sigma(f)^i f^{-p(m+i)}$
$(pdK)^{n+O(1)}$ terms		$(dK)^{n+O(1)}$ terms

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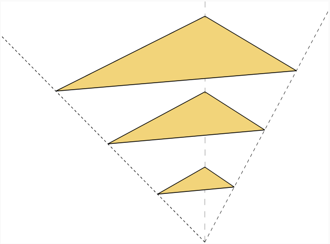

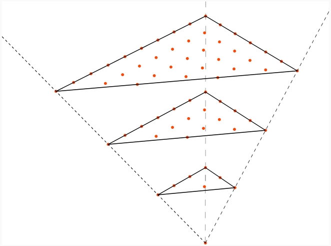

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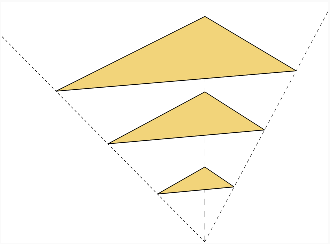

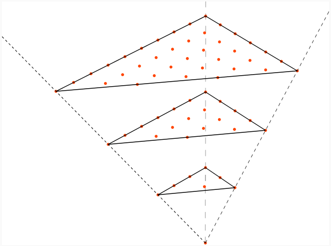

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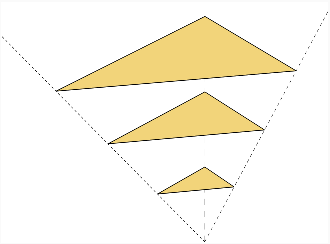

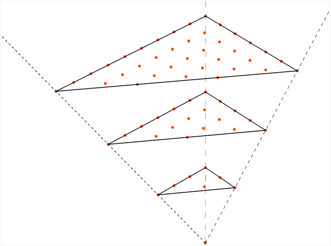

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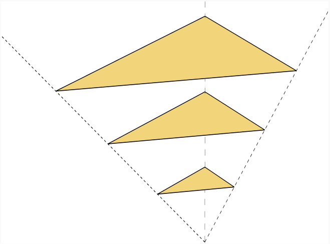

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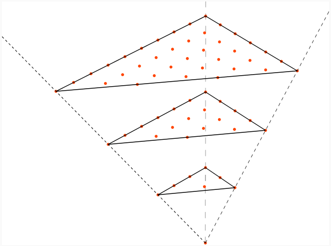

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Generic algorithm – C.–Harvey–Kedlaya

$$H^n_{\mathrm{dR}}(U) \xrightarrow[\mathrm{id}]{\sim} H^{\dagger,n}(U_{\mathbb{F}_p}) \circlearrowright \sigma$$

1. Compute $\left\{ \frac{x^\beta}{f^m}\omega \right\}_\beta$ a monomial basis for $H^n_{\mathrm{dR}}(U_{\mathbb{Q}_q})$
2. In $H^{\dagger,n}$ compute a **sparse** approximation for

$$\sigma\left(\frac{x^\beta}{f^m}\omega\right) \approx p^n \frac{x^{p\beta}}{f^{pm}} \sum_{i=0}^{N-1} \binom{-m}{i} \binom{m+N-1}{N-i-1} \sigma(f)^i f^{-p(m+i)}$$

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For large p, all the work is in step 3

Some Remarks

- **Complexity**
  First version of our new algorithm has complexity roughly

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- **Implementation**
  - Projective hypersurfaces (∼2014): C++ with NTL and Flint
    Soon available in Sage
  - Toric hypersurfaces: beta version in C++ with NTL

Some examples

Example: random dense K3 surface

$X \subset \mathbb{P}^3_{\mathbb{F}_p}$ given by

$$\begin{aligned}
&-9x^4 - 10x^3y - 9x^2y^2 + 2xy^3 - 7y^4 + 6x^3z + 9x^2yz - 2xy^2z + 3y^3z \\
&+ 8x^2z^2 + 6y^2z^2 + 2xz^3 + 7yz^3 + 9z^4 + 8x^3w + x^2yw - 8xy^2w - 7y^3w \\
&+ 9x^2zw - 9xyzw + 3y^2zw - xz^2w - 3yz^2w + z^3w - x^2w^2 - 4xyw^2 \\
&- 3xzw^2 + 8yzw^2 - 6z^2w^2 + 4xw^3 + 3yw^3 + 4zw^3 - 5w^4 = 0
\end{aligned}$$

For $p = 49999$, in `1h5m5s`, we obtain

$$\zeta_X(t) = ((1-t)(1-pt)(1-p^2t)Q(t))^{-1}$$

where

$$\begin{aligned}
pQ(t/p) =&(1-t)(p + 63115t + 14796t^2 + 42361t^3 + 49443t^4 \\
&+ 11718t^5 + 42046t^6 + 51501t^7 + 20534t^8 + 27146t^9 \\
&+ 38370t^{10} + 27146t^{11} + 20534t^{12} + \cdots + pt^{20})
\end{aligned}$$

Example: a quartic surface in the Dwork pencil

Consider the surface X in $\mathbb{P}^3_{\mathbb{F}_p}$ for $p = \texttt{4999999} = 5 \cdot 10^6 - 1$ given by

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 + x_0 x_1 x_2 x_3 = 0.$$

Using the old projective code in `100h30m` we compute that

$$\zeta_X(t) = \frac{1}{(1-t)(1-pt)(1-p^2t)R_1(pt)^3 R_2(pt)^6 S(t)}$$

where the “interesting” factor

$$S(t) = (1+pt)(1+5301514t+p^2t^2).$$

The polynomials R_1 and R_2 arise from the action of Frobenius on the Picard lattice; by a p-adic formula of de la Ossa–Kadir.

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$$x_0^4 + x_1^4 + x_2^4 + x_3^4 + x_0x_1x_2x_3 = 0.$$

Using the toric ~~old projective~~ code in `6m32s` ~~`100h30m`~~ we compute

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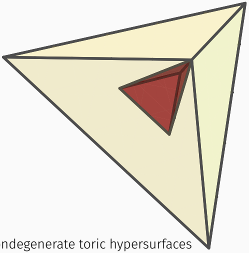
The polytope
of X is much smaller than the full simplex
($32/3 \approx 10.6$ vs $2/3 \approx 0.6$), as the monomials
defining X generate a sublattice of index 4^2 in $\mathbb{Z}^3$.



Example: a quintic threefold in the Dwork pencil

Consider the threefold X in $\mathbb{P}^4_{\mathbb{F}_p}$ for $p = 1000003$ given by

$$x_0^5 + \cdots + x_4^5 + x_0 x_1 x_2 x_3 x_5 = 0.$$

In 667s, we compute that

$$\zeta_X(t) = \frac{R_1(pt)^{20} R_2(pt)^{30} S(t)}{(1-t)(1-pt)(1-p^2 t)(1-p^3 t)}$$

where the "interesting" factor is

$$S(t) = 1 + 74132440T + 748796652370pT^2 + 74132440p^3T^3 + p^6T^4.$$

and R_1 and R_2 are the numerators of the zeta functions of certain curves (given by a formula of Candelas–de la Ossa–Rodriguez Villegas).

Using the old projective code, it would have taken around 37h.

Example: another family of K3 surfaces

Consider now the surface X in the weighted projective space $\mathbb{P}(8,5,4,3)_{\mathbb{F}_p}$ given by taking the closure of the affine surface

$$yz^5 + xz^4 + y^4 + z^4 + x^2 + 1 = 0.$$

For $p = 49999$, in **120s** we compute that

$$\zeta_X(t) = \frac{1}{(1-t)(1-pt)(1-p^2t)R(pt)S(t)}$$

where

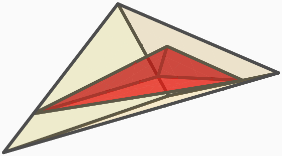
$$pS(p^{-1}t) = p - 14662t - 31559t^2 - 5620t^3 - 31559t^4 - 14662t^5 + pt^6.$$

This example is from Miles Reid's list of 95 families of nondegenerate toric surfaces which are K3 surfaces.

There is no other method that can handle dense surfaces in $\mathbb{P}(8,5,4,3)$ in this p range.

Consider the appropriate completion of the toric surface over $\mathbb{F}_p$ with $p = 49999$ given by

$$x^3y + y^4 + z^4 - 12xyz + 1 = 0.$$



In 61s, we compute that the "interesting" factor of $\zeta_X(t)$ is

$$1 - 9786t - 42243pt^2 + 35036p^2t^3 - 42243p^3t^4 - 9786p^4t^5 + p^6t^6.$$

In $\mathbb{P}^3$ this surface is degenerate, and would have taken us one hour to do the same computation with a dense model.

Example: a hypergeometric motive (also a K3 surface)

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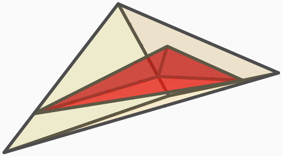

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We can confirm the linear term with Magma:

```
C2F2 := HypergeometricData([6,12], [1,1,1,2,3]);
EulerFactor(C2F2, 2^10 * 3^6, 49999: Degree:=1);
1 - 9786*$.1 + O($.1^2)
```

Example: another Calabi–Yau threefold

Let X be the closure in the weighted projective space $\mathbb{P}(10, 11, 16, 19, 21)_{\mathbb{F}_p}$ for $p = 49999$ of the affine threefold

$$y^7 + x^2zw + zyzw + y^2zw + z^3w + w^3 + xz + yz = 0.$$

In 401s, we we compute that the "interesting" factor of ζ_X is

$$1 + 6423186t + 2211095838pt^2 - 127485903944p^2t^3$$
$$+2211095838p^4t^4 + 6423186p^6T^5 + p^9T^6$$

By analogy with the Reid list, one can classify Calabi–Yau threefolds arising as hypersurfaces in weighted projective spaces; there are 7555 such families. See `http://hep.itp.tuwien.ac.at/~kreuzer/CY/`.

Other possible versions

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  These have not yet been implemented and we still need to write the paper...