Computing zeta functions of nondegenerate hypersurfaces in toric varieties

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May 16th, 2018

Presented at ICERM, Birational Geometry and Arithmetic
Joint work with David Harvey (UNSW) and Kiran Kedlaya (UCSD)

Slides available at `edgarcosta.org` under Research

Motivation

Riemann zeta function

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} \cdots$$
$$= \frac{1}{1-2^{-s}} \cdot \frac{1}{1-3^{-s}} \cdot \frac{1}{1-5^{-s}} \cdots$$

- One of the most famous examples of a global zeta function
- Together with the functional equation

$$\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s) = \xi(1-s)$$

  encodes a lot of the arithmetic information of $\mathbb{Z}$.
  e.g.: Zeros of $\zeta(s) \rightsquigarrow$ precise prime distribution
- $\zeta(s)$ still keeps secret many of its properties

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Hasse and Weil generalized an analog of $\zeta(s)$ for algebraic varieties

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- What arithmetic properties of X can we read from $\zeta_{X_p}(s)$?
- $\zeta_{X_p}(t)$ obeys a functional equation and satisfies the Riemann hypothesis!
- What about $\zeta_X(s)$?

Elliptic curves

E an elliptic curve over $\mathbb{Q}$

$$\zeta_E(s) := \prod_p \zeta_{E_p}(p^{-s}) \quad \text{and} \quad \zeta_{E_p}(t) = \frac{L_p(t)}{(1-t)(1-pt)}$$

$$L_p(t) = \begin{cases} 1 - a_p t + pt^2, & \text{good reduction}, a_p = p + 1 - \#E_p(\mathbb{F}_p) \\ 1 \pm t, & \text{non-split/split multiplicative reduction;} \\ 1 & \text{additive reduction;} \end{cases}$$

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- $a_p \rightsquigarrow$ arithmetic information about $E_p \rightsquigarrow E$.
- Modularity theorem $\implies$ L_E satisfies a functional equation
- Birch–Swinnerton-Dyer conjecture predicts $\operatorname{ord}_{s=1} L_E(s) = \operatorname{rk}(E)$.

$\zeta(s)$ vs $\zeta_X(s)$

We always expect $\zeta_X(s)$ to satisfy a functional equation.

- zero-dimensional varieties (number fields) ✓
- elliptic curves over $\mathbb{Q}$ ✓
- genus 2 curves ?

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- easy to explicitly write down $\zeta(s)$
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Problem

Given an *explicit* description of X, compute

$$\zeta_{X_p}(t) := exp\left(\sum_{i\geq 0} \#X_p(\mathbb{F}_{p^i})\frac{t^i}{i}\right) \in \mathbb{Q}(t)$$

The zeta function problem

Let X be a smooth variety over a finite field $\mathbb{F}_q$ of characteristic p, consider

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Compute ζ_X from an *explicit* description of X.

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This approach is only practical for very few classes of varieties, e.g., low genus curves and p small.

“Real life” applications

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- Testing the speciality of a cubic fourfold
- Computing L-functions and their special values, e.g.:
  - Birch–Swinnerton-Dyer conjecture $\rightsquigarrow \mathrm{rk}(A)$
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- Arithmetic statistics
  - Sato–Tate
  - Lang–Trotter

Common Approaches

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- p-adic: based on Monsky–Washnitzer cohomology

Today

New p-adic method to compute $\zeta_X(t)$ that achieves a striking balance between **practicality** and **generality**.

Outline

Toric hypersurfaces

p-adic Cohomology

Some examples

Toric hypersurfaces

Toy example, the Projective space

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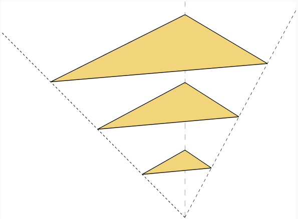
- We can think of $P_d := R[d\Delta \cap \mathbb{Z}^n]$, where Δ is the standard simplex.
- Idea: generalize Δ to be any polytope.



Toric hypersurfaces

- $f = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha x^\alpha \in R[x_1^\pm, \ldots, x_n^\pm]$ a Laurent polynomial
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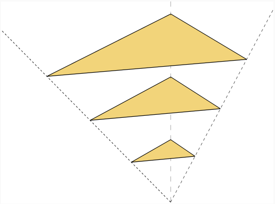
$$P_\Delta := \bigoplus_{d \geq 0} P_d, \quad P_d := R[x^\alpha : \alpha \in d\Delta \cap \mathbb{Z}^n]$$

$$\mathbb{P}_\Delta := \mathrm{Proj}\, P_\Delta$$

$$X_f := \mathrm{Proj}\, P_\Delta/(f) \subset \mathbb{P}_\Delta$$



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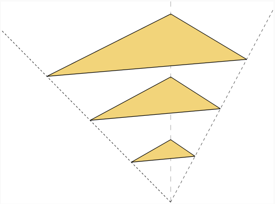

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Examples

Δ	X_Δ
$\mathrm{Conv}(0, e_1, \ldots, e_n)$	$\mathbb{P}^n$
$\mathrm{Conv}(0, e_1, \ell e_2, \ldots, \ell e_n)$	$\mathbb{P}^n(\ell, 1, \ldots, 1)$
$\mathrm{Conv}(0, e_1, e_2, e_1 + e_2) = [0, 1]^2$	$\mathbb{P}^1 \times \mathbb{P}^1$

Toric hypersurfaces are everywhere

Vertices of Δ	Resulting hypersurface
$0, de_1, de_2$	Smooth plane curve of genus $\binom{d-1}{2}$
$0, (2g+1)e_1, 2e_2$	Odd hyperelliptic curve of genus g
$0, ae_1, be_2$	$C_{a,b}$-curve
$0, 4e_1, 4e_2, 4e_3$	Quartic K3 surface
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K3 surfaces can arise as hypersurfaces:

- in $\mathbb{P}^3$, as a quartic surface;
- in 95 weighed projective spaces;
- in **4319** toric varieties.

Keeping our eyes on the prize

Given

$$f = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha x^\alpha \in \mathbb{F}_q[x_1^{\pm}, \ldots, x_n^{\pm}]$$

efficiently compute

$$\begin{aligned}\zeta_X(t) &:= exp\left(\sum_{i \geq 1} \#X(\mathbb{F}_{q^i}) \frac{t^i}{i}\right) \\ &= \prod_i \det\bigl(1 - t\,\mathrm{Frob}\,|H^i_{\mathrm{et}}(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell)\bigr)^{(-1)^{i+1}} \in \mathbb{Q}(t),\end{aligned}$$

where $X := \mathrm{Proj}\, P_\Delta/(f) \subset \mathbb{P}_\Delta$

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We will need a bit more, we will **nondegeneracy**.

Nondegenerate toric hypersurfaces

Geometric definition

An hypersurface is **nondegenerate** if the cross-section by any bounding hyperplane (in any dimension) are all smooth in their respective tori.

Equivalently, if for every face $\sigma \subseteq \Delta$, f restricted to the torus associated to σ is nonsingular of codimension 1.

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Example

Let C be a plane curve in $\mathbb{P}^2$, then C is nondegenerate if:

- C does not pass through the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$;
- C intersects the coordinate axes $x = 0$, $y = 0$, $z = 0$ transversally;
- C is smooth on the complement of the coordinate axes.

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In terms of ideals, $\operatorname{rad} \left\langle x\frac{\partial}{\partial x}f, y\frac{\partial}{\partial y}f, z\frac{\partial}{\partial z}f, f \right\rangle = \langle x, y, z \rangle$

p-adic Cohomology

Goal

Setup

- $f = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha x^\alpha \in \mathbb{F}_q[x_1^\pm, \ldots, x_n^\pm]$
- $X := \operatorname{Proj} P_\Delta/(f) \subset \mathbb{P}_\Delta$ a nondegenerate hypersurface

Goal

Compute

$$\begin{aligned}\zeta_X(t) &:= exp\left(\sum_{i\geq 1} \#X(\mathbb{F}_{q^i}) t^i / i\right) \\ &= \prod_i \det\left(1 - t \operatorname{Frob} | H^i_{\mathrm{et}}(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell)\right)^{(-1)^{i+1}} \\ &= Q(t)^{(-1)^n} \zeta_{\mathbb{P}_\Delta}(t),\end{aligned}$$

where $\quad Q(t) := \det(1 - t \operatorname{Frob} | PH^{n-1}_{\mathrm{et}}(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell)) \in 1 + \mathbb{Z}[t]$

Master plan

Setup

- $f = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha x^\alpha \in \mathbb{F}_q[x_1^\pm, \ldots, x_n^\pm]$
- $X := \operatorname{Proj} P_\Delta/(f) \subset \mathbb{P}_\Delta$ a nondegenerate hypersurface
- $\sigma :=$ p-th power Frobenius map

Goal

Compute the matrix representing the action of σ in $PH^{n-1}_{\mathrm{rig}}(X)$ with enough of p-adic precision to deduce

$$Q(t) = \det(1 - q^{-1} t \operatorname{Frob} | PH^{n-1}_{\mathrm{rig}}(X)) \in 1 + \mathbb{Z}[t].$$

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- $\sigma :=$ p-th power Frobenius map

Goal

Compute the matrix representing the action of σ in $PH^{n-1}_{\text{rig}}(X)$ with enough of p-adic precision to deduce

$$Q(t) = \det(1 - q^{-1} t \operatorname{Frob} | PH^{n-1}_{\text{rig}}(X)) \in 1 + \mathbb{Z}[t].$$

Instead, of working with rigid cohomology, we will work with the Monsky–Washnitzer cohomology $PH^{\dagger, n-1}(X)$

Master plan

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de Rham cohomology
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cohomology relations
+
commutative algebra

$\Longrightarrow$

basis for $PH^{n-1}_{\mathrm{dR}}(X_{\mathbb{Q}_q}) = \left\{ x^{\beta}\omega/f^{i} \right\}_{\beta}$
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reduction algorithm

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Note: Originally for smooth hypersurfaces in the projective space.

A sparse representation of Frobenius

Unfortunately, the truncation of the series expansion to K terms

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Schematically

Abbott–Kedlaya–Roe	vs	C.–Harvey–Kedlaya
$\sum_{i=0}^{K-1} \binom{-m}{i} \left(\frac{\sigma(f)-f^p}{f^p}\right)^i$		$\sum_{i=0}^{K-1} \binom{-m}{i} \binom{m+K-1}{K-i-1} \sigma(f)^i f^{-p(m+i)}$
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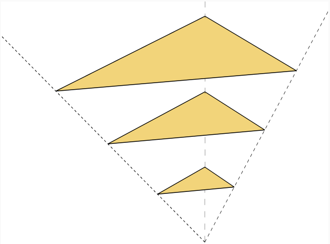

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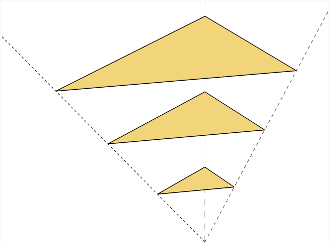

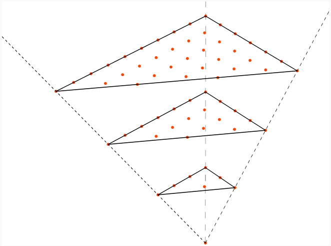

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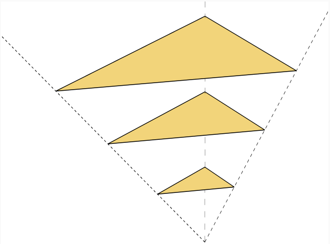

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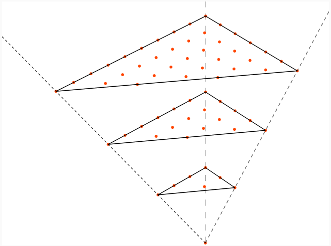

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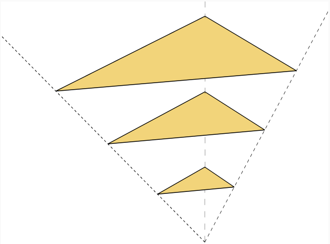

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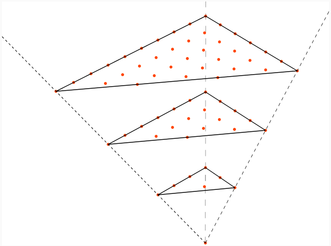

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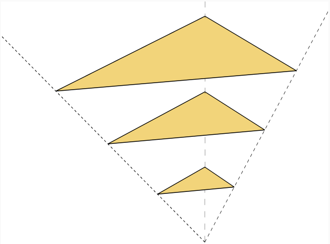

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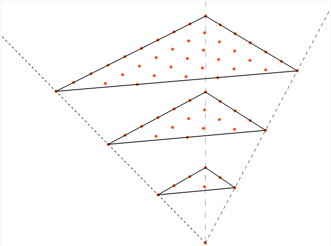

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For large p, all the work is in step 3

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  - Toric hypersurfaces: beta version in C++ with NTL

Some examples

Example: K3 surface in the Dwork pencil

Consider the projective quartic surface X in $\mathbb{P}^3_{\mathbb{F}_p}$ given by

$$x^4 + y^4 + z^4 + w^4 + \lambda xyzw = 0.$$

For $\lambda = 1$ and $p = 2^{20} - 3$, using the old projective code in `22h7m` we compute that

$$\zeta_X(t)^{-1} = (1 - t)(1 - pt)^{16}(1 + pt)^3(1 - p^2 t)Q(t),$$

where the "interesting" factor is

$$Q(t) = (1 + pt)(1 - 1688538t + p^2 t^2).$$

The polynomials R_1 and R_2 arise from the action of Frobenius on the Picard lattice; by a p-adic formula of de la Ossa–Kadir.

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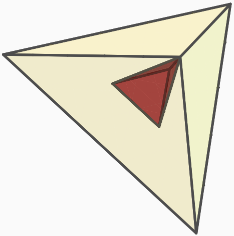
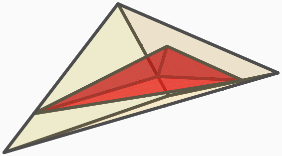
The defining monomials of X generate a sublattice of index 4^2 in $\mathbb{Z}^3$, and we can work "in" that sublattice, by using

$$x^4y^{-1}z^{-1} + \lambda x + y + z + 1 = 0$$

which has a polytope much smaller than the full simplex ($32/3 \approx 10.6$ vs $2/3 \approx 0.6$).



Example: a hypergeometric motive (also a K3 surface)



Consider the appropriate completion of the toric surface over $\mathbb{F}_p$ with $p = 2^{15} - 19$ given by

$$x^3 y + y^4 + z^4 - 12xyz + 1 = 0.$$

In `4s`, we compute that the "interesting" factor of $\zeta_X(t)$ is (up to rescaling)

$$pQ(t/p) = p + 20508t^1 - 18468t^2 - 26378t^3 - 18468t^4 + 20508t^5 + pt^6.$$

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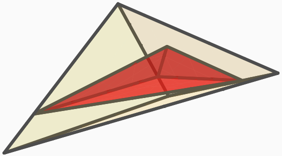

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We can confirm the linear term with Magma:

```
C2F2 := HypergeometricData([6,12], [1,1,1,2,3]);
EulerFactor(C2F2, 2^10 * 3^6, 2^15-19: Degree:=1);
1 + 20508*$.1 + O($.1^2)
```

Example: a K3 surface in a non weighted projective space

Consider the surface X defined as the closure (in $\mathbb{P}_\Delta$) of the affine surface defined by the Laurent polynomial

$$3x + y + z + x^{-2}y^2z + x^3y^{-6}z^{-2} + 3x^{-2}y^{-1}z^{-2} - 2 - x^{-1}y - y^{-1}z^{-1} - x^2y^{-4}z^{-1} - xy^{-3}z^{-1}.$$

The Hodge numbers of $PH^2(X)$ are $(1, 14, 1)$. For $p = 2^{15} - 19$, in `6m20s` we obtain the "interesting" factor of $\zeta_X(t)$:

$$pQ(t/p) = (1-t)\cdot(1+t)\cdot(p + 33305t^1 + 1564t^2 - 14296t^3 - 11865t^4 + 5107t^5 + 27955t^6 + 25963t^7 + 27955t^8 + 5107t^9 - 11865t^{10} - 14296t^{11} + 1564t^{12} + 33305t^{13} + pt^{14}).$$

We know of no previous algorithm that can compute $\zeta_X(t)$ for p in this range!

Example: random dense K3 surface

$X \subset \mathbb{P}^3_{\mathbb{F}_p}$ given by

$$-9x^4 - 10x^3y - 9x^2y^2 + 2xy^3 - 7y^4 + 6x^3z + 9x^2yz - 2xy^2z + 3y^3z$$
$$+ 8x^2z^2 + 6y^2z^2 + 2xz^3 + 7yz^3 + 9z^4 + 8x^3w + x^2yw - 8xy^2w - 7y^3w$$
$$+ 9x^2zw - 9xyzw + 3y^2zw - xz^2w - 3yz^2w + z^3w - x^2w^2 - 4xyw^2$$
$$- 3xzw^2 + 8yzw^2 - 6z^2w^2 + 4xw^3 + 3yw^3 + 4zw^3 - 5w^4 = 0$$

For $p = 2^{15} - 19$, in `38m27s`, we obtain

$$\zeta_X(t) = ((1-t)(1-pt)(1-p^2t)Q(t))^{-1}$$

where

$$pQ(t/p) = (t+1)\big(p - 53159t^1 + 10023t^2 - 3204t^3 + 49736t^4 - 56338t^5$$
$$+ 43086t^6 - 48180t^7 + 44512t^8 - 42681t^9 + 47794t^{10}$$
$$- 42681t^{11} + 44512t^{12} - 48180t^{13} + 43086t^{14} - 56338t^{15}$$
$$+ 49736t^{16} - 3204t^{17} + 10023t^{18} - 53159t^{19} + pt^{20}\big)$$

Old implementation takes roughly the same time.

Example: a quintic threefold in the Dwork pencil

Consider the threefold X in $\mathbb{P}^4_{\mathbb{F}_p}$ for $p = 2^{20} - 3$ given by

$$x_0^5 + \cdots + x_4^5 + x_0 x_1 x_2 x_3 x_5 = 0.$$

In `11m18s`, we compute that

$$\zeta_X(t) = \frac{R_1(pt)^{20} R_2(pt)^{30} S(t)}{(1-t)(1-pt)(1-p^2 t)(1-p^3 t)}$$

where the "interesting" factor is

$$S(t) = 1 + 74132440T + 748796652370pT^2 + 74132440p^3T^3 + p^6T^4.$$

and R_1 and R_2 are the numerators of the zeta functions of certain curves (given by a formula of Candelas–de la Ossa–Rodriguez Villegas).

Using the old projective code, we extrapolate it would have taken us at least 120 days.

Example: a Calabi–Yau 3fold in a non weighted projective space

Let X be the closure (in $\mathbb{P}_\Delta$) of the affine threefold

$$xyz^2w^3 + x + y + z - 1 + y^{-1}z^{-1} + x^{-2}y^{-1}z^{-2}w^{-3} = 0.$$

For $p = 2^{20} - 3$, in `1h15m`, we computed the "interesting" factor of $\zeta_X(t)$

$$(1+718pt+p^3t^2)(1+1188466826t+1915150034310pt^2+1188466826p^3t^3+p^6t^4).$$

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By analogy with the Reid's list, Calabi–Yau threefolds can arise as hypersurfaces in:

- 7555 weighted projective spaces;
- 473,800,776 toric varieties.

See `http://hep.itp.tuwien.ac.at/~kreuzer/CY/`.

Example: a dense Cubic fourfold

$$\begin{aligned} x_0^2x_1 + x_0x_1^2 + x_1^2x_2 + x_0x_2^2 + 4x_0^2x_3 + x_1^2x_3 \\ + 8x_0x_2x_3 + 2x_1x_2x_3 + 2x_2^2x_3 + 4x_0x_3^2 + x_1x_3^2 + x_3^3 + 8x_0x_1x_4 \\ + x_1^2x_4 + 4x_1x_2x_4 + x_2^2x_4 + 8x_0x_3x_4 + 2x_2x_3x_4 + 8x_0x_4^2 \\ + x_1x_4^2 + 2x_3x_4^2 + x_4^3 + 2x_0^2x_5 + x_1^2x_5 + x_1x_2x_5 + x_2^2x_5 \\ + 8x_0x_3x_5 + x_1x_3x_5 + x_3^2x_5 + 4x_0x_4x_5 + 3x_3x_4x_5 + 2x_0x_5^2 + x_4x_5^2. \end{aligned}$$

For $p = 23$, in `22h52m`, we computed $\zeta_X(t)$ using a a **fully dense** nondegenerate model, obtained by random change of variables in $\mathbb{P}^5$. And we concluded that $\rho(X) = 3$ (one extra class over $\mathbb{F}_p$ and another one over $\mathbb{F}_{p^2}$).

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For $p = 113$ the running time was `26h34m` and for $p = 499$ it was `33h47m`.

Most of the time is spent setting up and solving the initial linear algebra problems.

Other possible versions

- **Space-time tradeoff**
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These have not yet been implemented and we still need to write the paper...